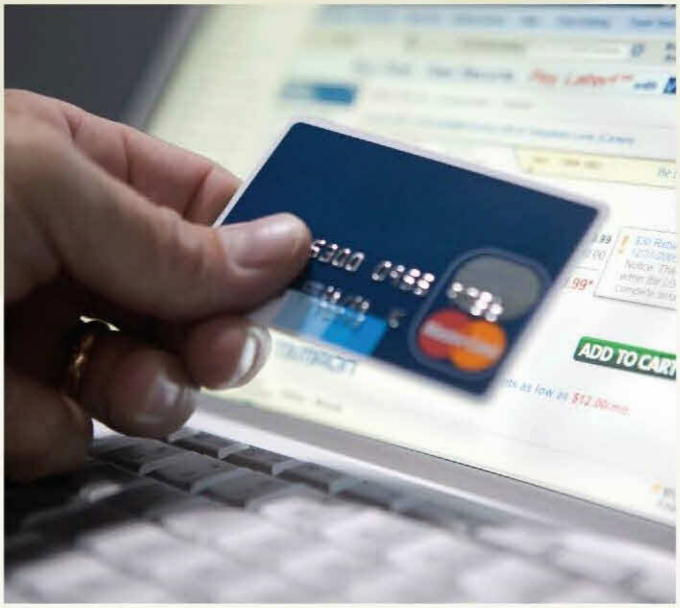
1

Functions and Linear Models

Case Study Modeling Spending on Internet Advertising

You are the new director of *Impact Advertising Inc.'s* Internet division, which has enjoyed a steady 0.25% of the Internet advertising market. You have drawn up an ambitious proposal to expand your division in light of your anticipation that Internet advertising will continue to skyrocket. The VP in charge of Financial Affairs feels that current projections (based on a linear model) do not warrant the level of expansion you propose. **How can you persuade the VP that those projections do not fit the data convincingly?**



Jeff Titcomb/Photographer's Choice / Getty Images

Introduction

To analyze recent trends in spending on Internet advertising and to make reasonable projections, we need a mathematical model of this spending. Where do we start? To apply mathematics to real-world situations like this, we need a good understanding of basic mathematical concepts. Perhaps the most fundamental of these concepts is that of a function: a relationship that shows how one quantity depends on another. Functions may be described numerically and, often, algebraically. They can also be described graphically—a viewpoint that is extremely useful.

The simplest functions—the ones with the simplest formulas and the simplest graphs—are linear functions. Because of their simplicity, they are also among the most useful functions and can often be used to model real-world situations, at least over short periods of time. In discussing linear functions, we will meet the concepts of slope and rate of change, which are the starting point of the mathematics of change.

In the last section of this chapter, we discuss *simple linear regression*: construction of linear functions that best fit given collections of data. Regression is used extensively in applied mathematics, statistics, and quantitative methods in business. The inclusion of regression utilities in computer spreadsheets like Excel® makes this powerful mathematical tool readily available for anyone to use.

algebra Review
For this chapter, you should be familiar with real numbers and intervals. To review this material, see **Chapter 0.**

1.1 Functions from the Numerical, Algebraic, and Graphical Viewpoints

The following table gives the approximate number of Facebook users at various times since its establishment early in 2004.[1]

Year *t* (Since start of 2004)	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5
Facebook Members *n* (Millions)	0	0.5	1	2	5.5	7	12	30	58	80

Let's write $n(0)$ for the number of members (in millions) at time $t = 0$, $n(0.5)$ for the number at time $t = 0.5$, and so on. (We read $n(0)$ as "n of 0.") Thus, $n(0) = 0$, $n(0.5) = 0.5$, $n(1) = 1$, $n(1.5) = 2, \ldots, n(4.5) = 80$. In general, we write $n(t)$ for the number of members (in millions) at time t. We call n a **function** of the variable t, meaning that for each value of t between 0 and 4.5, n gives us a single corresponding number $n(t)$ (the number of members at that time).

In general, we think of a function as a way of producing new objects from old ones. The functions we deal with in this text produce new numbers from old numbers. The numbers we have in mind are the *real* numbers, including not only positive and negative integers and fractions but also numbers like $\sqrt{2}$ or π. (See Appendix A for more on real numbers.) For this reason, the functions we use are called **real-valued functions of a real variable**. For example, the function n takes the year since the start of 2004 as input and returns the number of Facebook members as output (Figure 1).

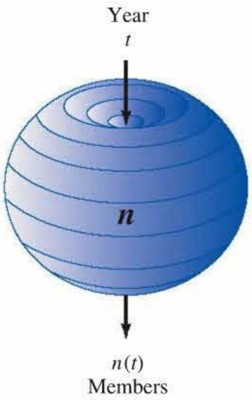


Figure 1

---

[1] Sources: www.facebook.com, www.insidehighered.com (Some data are interpolated.)

The variable t is called the **independent** variable, while n is called the **dependent variable** as its value depends on t. A function may be specified in several different ways. Here, we have specified the function n **numerically** by giving the values of the function for a number of values of the independent variable, as in the preceding table.

Q: *For which values of t does it make sense to ask for n(t)? In other words, for which years t is the function n defined?*

A: Because $n(t)$ refers to the number of members from the start of 2004 to the middle of 2008, $n(t)$ is defined when t is any number between 0 and 4.5, that is, when $0 \leq t \leq 4.5$. Using interval notation (see Chapter 0), we can say that $n(t)$ is defined when t is in the interval [0, 4.5].

The set of values of the independent variable for which a function is defined is called its **domain** and is a necessary part of the definition of the function. Notice that the preceding table gives the value of $n(t)$ at only some of the infinitely many possible values in the domain [0, 4.5]. The domain of a function is not always specified explicitly; if no domain is specified for the function f, we take the domain to be the largest set of numbers x for which $f(x)$ makes sense. This "largest possible domain" is sometimes called the **natural domain**.

The previous Facebook data can also be represented on a graph by plotting the given pairs of numbers $(t, n(t))$ in the xy-plane. (See Figure 2.) We have connected successive points by line segments. In general, the **graph** of a function f consists of all points $(x, f(x))$ in the plane with x in the domain of f.

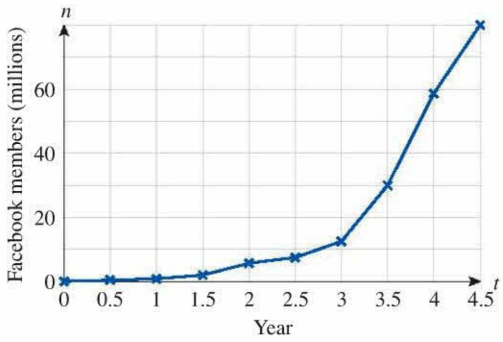


Figure 2

In Figure 2 we specified the function n **graphically** by using a graph to display its values. Suppose now that we had only the graph without the table of data. We could use the graph to find approximate values of n. For instance, to find $n(3)$ from the graph, we do the following:

1. Find the desired value of t at the bottom of the graph ($t = 3$ in this case).
2. Estimate the height (n-coordinate) of the corresponding point on the graph (around 12 in this case).

Thus, $n(3) \approx 12$ million members.*

* **NOTE** In a graphically defined function, we can never know the *y* coordinates of points exactly; no matter how accurately a graph is drawn, we can obtain only *approximate* values of the coordinates of points. That is why we have been using the word *estimate* rather than *calculate* and why we say "$n(3) \approx 12$" rather than "$n(3) = 12$."

In some cases we may be able to use an algebraic formula to calculate the function, and we say that the function is specified **algebraically**. These are not the only ways in which a function can be specified; for instance, it could also be specified **verbally**, as in "Let $n(t)$ be the number of Facebook members, in millions, t years since the start of 2004."* Notice that any function can be represented graphically by plotting the points $(x, f(x))$ for a number of values of x in its domain.

Here is a summary of the terms we have just introduced.

* **NOTE** Specifying a function verbally in this way is useful for understanding what the function is doing, but it gives no numerical information.

Functions

A **real-valued function f of a real-valued variable x** assigns to each real number x in a specified set of numbers, called the **domain** of f, a unique real number $f(x)$, read "f of x." The variable x is called the **independent variable**, and f is called the **dependent variable**. A function is usually specified **numerically** using a table of values, **graphically** using a graph, or **algebraically** using a formula. The **graph of a function** consists of all points $(x, f(x))$ in the plane with x in the domain of f.

Quick Examples

1. **A function specified numerically:** Take $f(t)$ to be the amount of freon (in tons) produced in developing countries in year t since 2000, represented by the following table:

t (Year Since 2000)	$f(t)$ (Tons of Freon 22)
0	100
2	140
4	200
6	270
8	400
10	590

Source: *New York Times*, February 23, 2007, p. C1.

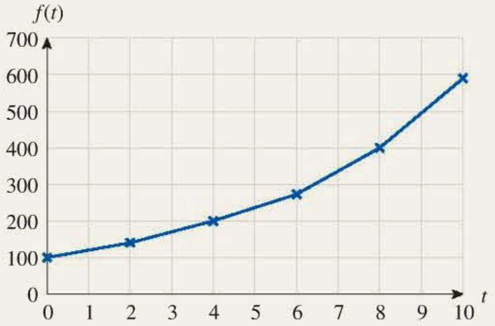
The domain of f is [0, 10], the independent variable is t, the number of years since 2000, and the dependent variable is f, the number of tons of freon produced in a year in developing countries. Some values of f are:

$f(0) = 100$ 100 tons of freon were produced in developing countries in 2000.

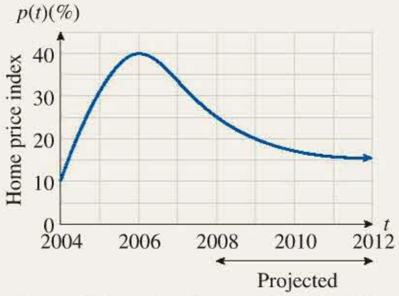
$f(6) = 270$ 270 tons of freon were produced in developing countries in 2006.

$f(10) = 590$ 590 tons of freon were produced in developing countries in 2010.

Graph of f: Plotting the pairs $(t, f(t))$ gives the following graph:



2. **A function specified graphically:** Take $p(t)$ to be the home price index as a percentage change from 2003 in year t, represented by the following graph:



S&P/Case-Shiller Home Price Index. Source: Standard & Poors/Bloomberg Financial Markets/*New York Times*, September 29, 2007, p. C3. Projection is the authors'.*

The domain of p is [2004, 2012], the independent variable is the the year t, and the dependent variable is the percentage p above the 2003 value. Some values of p are:

$p(2004) \approx 10$ In 2004 the index was about 10% above the 2003 value.

$p(2006) \approx 40$ In 2006 the index was about 40% above the 2003 value.

$p(2009) \approx 20$ In 2009 the index was about 20% above the 2003 value.

3. **A function specified algebraically:** Let $f(x) = \frac{1}{x}$. The function f is specified algebraically. The independent variable is x and the dependent variable is f. The natural domain of f consists of all real numbers except zero because $f(x)$ makes sense for all values of x other than $x = 0$. Some specific values of f are

$$f(2) = \frac{1}{2} \qquad f(3) = \frac{1}{3} \qquad f(-1) = \frac{1}{-1} = -1$$

$f(0)$ is not defined because 0 is not in the domain of f.

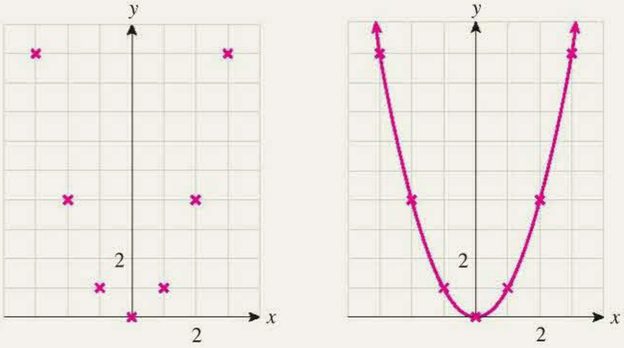
* **NOTE** *added in July 2009:* Our projection turned out to be wrong: The index fell further than we anticipated and wound up negative! This error illustrates the pitfalls of *extrapolation*—a point we will discuss after Example 1.

4. The graph of a function: Let $f(x) = x^2$, with domain the set of all real numbers. To draw the graph of f, first choose some convenient values of x in the domain and compute the corresponding y-coordinates $f(x)$:

x	-3	-2	-1	0	1	2	3
$f(x) = x^2$	9	4	1	0	1	4	9

Plotting these points $(x, f(x))$ gives the picture on the left, suggesting the graph on the right.*

*** NOTE** If you plot more points, you will find that they lie on a smooth curve as shown. That is why we did not use line segments to connect the points.



(This particular curve happens to be called a **parabola**, and its lowest point, at the origin, is called its **vertex**.)


Bartomeu Amengual/Index Stock Imagery / PhotoLibrary

EXAMPLE 1 iPod Sales

The number of iPods sold by Apple Inc. each year from 2004 through 2007 can be approximated by

$$f(x) = -x^2 + 20x + 3 \text{ million iPods} \qquad (0 \le x \le 3)$$

where x is the number of years since 2004.*

a. What is the domain of f? Compute $f(0)$, $f(1)$, $f(2)$, and $f(3)$. What do these answers tell you about iPod sales? Is $f(-1)$ defined?

b. Compute $f(a)$, $f(-b)$, and $f(a + h)$ assuming that the quantities a, $-b$, and $a + h$ are in the domain of f.

c. Sketch the graph of f. Does the shape of the curve suggest that iPod sales were accelerating or decelerating?

Solution

a. The domain of f is the set of numbers x with $0 \le x \le 3$—that is, the interval [0, 3]. If we substitute 0 for x in the formula for $f(x)$, we get

$$f(0) = -(0)^2 + 20(0) + 3 = 3$$ In 2004 approximately 3 million iPods were sold.

---

* Source for data: Apple quarterly earnings reports at www.apple.com/investor/.

so $f(0) = 3$. Similarly,

$$f(1) = -(1)^2 + 20(1) + 3 = 22$$ In 2005 approximately 22 million iPods were sold.

$$f(2) = -(2)^2 + 20(2) + 3 = 39$$ In 2006 approximately 39 million iPods were sold.

$$f(2) = -(3)^2 + 20(3) + 3 = 54$$ In 2007 approximately 54 million iPods were sold.

Because -1 is not in the domain of f, $f(-1)$ is not defined.

b. To find $f(a)$ we substitute a for x in the formula for $f(x)$ to get

$$f(a) = -a^2 + 20a + 3.$$ Substitute a for x.

Similarly,

$$f(-b) = -(-b)^2 + 20(-b) + 3$$ Substitute $-b$ for x.

$$= -b^2 - 20b + 3$$ $(-b)^2 = b^2$

$$f(a + h) = -(a + h)^2 + 20(a + h) + 3$$ Substitute $(a + h)$ for x.

$$= -(a^2 + 2ah + h^2) + 20a + 20h + 3$$ Expand.

$$= -a^2 - 2ah - h^2 + 20a + 20h + 3$$

Note how we placed parentheses around the quantities at which we evaluated the function. If we omitted any of these parentheses, we would likely get errors.

$$f(-b) = -(-b)^2 + 20(-b) + 3 \checkmark$$ **NOT** $--b^2 + 20(-b) + 3$ ×

$$f(a + h) = -(a + h)^2 + 20(a + h) + 3 \checkmark$$ **NOT** $-a + h^2 + 20a + h + 3$ ×

c. To draw the graph of f, we plot points of the form $(x, f(x))$ for several values of x in the domain of f. Let us use the values we computed in part (a):

x	0	1	2	3
$f(x) = -x^2 + 20x + 3$	3	22	39	54

Graphing these points gives the graph shown in Figure 3(a), suggesting the curve shown in Figure 3(b).

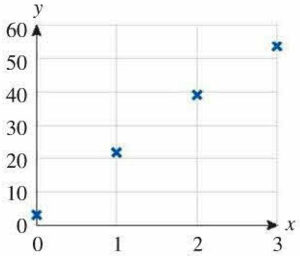


Figure 3(a)

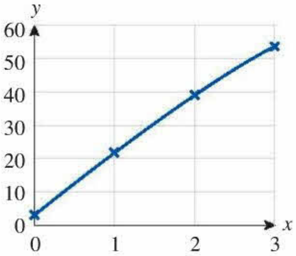


Figure 3(b)

The graph is becoming less steep as we move from left to right, suggesting that iPod sales were decelerating slightly. (This concept will be made more precise in Chapter 4.)

using Technology

See the Technology Guides at the end of the chapter for detailed instructions on how to obtain the table of values and graph in Example 1 using a TI-83/84 Plus or Excel. Here is an outline:

TI-83/84 Plus

Table of values:
`Y1=-X^2+20X+3`
`2ND` `TABLE`.
Graph: `WINDOW`;
Xmin = 0, Xmax = 3
`ZOOM` `0`.
[More details on page 110.]

Excel

Table of values: Headings x and $f(x)$ in A1–B1; t-values 0, 1, 2, 3 in A2–A5.
`=-1*A2^2+20*A2+3`
in B2; copy down through B5.
Graph: Highlight A1 through B5 and insert a Scatter chart.
[More details on page 116.]

Web Site

www.FiniteandCalc.org
Go to the Function Evaluator and Grapher under Online Utilities and enter
`-x^2+20x+3`
for y_1. To obtain a table of values, enter the x-values 0, 1, 2, 3 in the Evaluator box and press "Evaluate."
Graph: Set Xmin = 0 and Xmax = 3, and press "Plot Graphs."

➡ **Before we go on...** The following table compares the value of f in Example 1 with the actual sales figures:

Year x	0	1	2	3
$f(x) = -x^2 + 20x + 3$	3	22	39	54
Actual iPod Sales (millions)	4	22	39	52

The actual figures are stated here only for integer values of x; for instance, $x = 2$ gives the sales for the year ending December 2006. But what were, for instance, the sales for the year ending June 2007 ($x = 2.5$)? This is where our formula comes in handy: We can use the formula for f to **interpolate**; that is, to find sales at values of x between those that are stated:

$$f(2.5) = -(2.5)^2 + 20(2.5) + 3 = 46.75 \approx 47 \text{ million iPods}$$

We can also use the formula to **extrapolate**; that is, to predict sales at values of x *outside* the domain—say, for $x = 3.5$ (the year ending June 2008):

$$f(3.5) = -(3.5)^2 + 20(3.5) + 3 = 60.75 \approx 61 \text{ million iPods}$$

As a general rule, extrapolation is far less reliable than interpolation: Predicting the future from current data is difficult, especially given the vagaries of the marketplace.

We call the algebraic function f an **algebraic model** of iPod sales because it uses an algebraic formula to model—or mathematically represent (approximately)—the annual sales. The particular kind of algebraic model we used is called a **quadratic model**. (See the end of this section for the names of some commonly used models.) ■

Note **Equation and Function Notation**

Instead of using *function notation*

$$f(x) = -x^2 + 20x + 3 \qquad \text{Function notation}$$

we could use *equation notation*

$$y = -x^2 + 20x + 3 \qquad \text{Equation notation}$$

(the choice of the letter y is a convention) and we say that "y is a function of x." When we write a function in this form, the variable x is the independent variable and y is the dependent variable.

We could also write the above function as $f = -x^2 + 20x + 3$, in which case the dependent variable would be f. ■

Look again at the graph of the number of Facebook users in Figure 2. From year 0 through year 3, the membership appears to curve gently upward, but then increases quite dramatically from year 3 to year 4. This behavior can be modeled by using two different functions: one for the interval [0, 3] and another for the interval [3, 4]. (See Figure 4.)

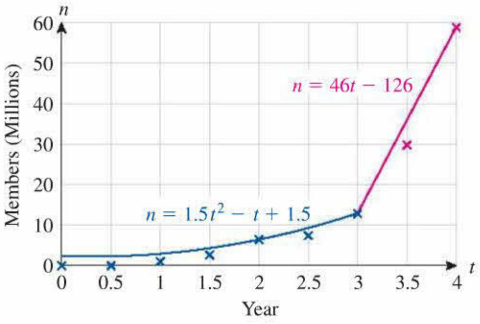


Figure 4

A function specified by two or more different formulas like this is called a **piecewise-defined function**.

EXAMPLE 2 A Piecewise-Defined Function: Facebook Membership

The number $n(t)$ of Facebook members can be approximated by the following function of time t in years ($t = 0$ represents January 2004):*

$$n(t) = \begin{cases} 1.5t^2 - t + 1.5 & \text{if } 0 \le t \le 3 \\ 46t - 126 & \text{if } 3 < t \le 5 \end{cases} \qquad \text{million members}$$

(Its graph is shown in Figure 4.) What was the approximate membership of Facebook in January 2005, January 2007, and June 2008? Sketch the graph of n by plotting several points.

Solution We evaluate the given function at the corresponding values of t:

Jan. 2005 ($t = 1$): $n(1) = 1.5(1)^2 - 1 + 1.5 = 2$ Use the first formula because $0 \le t \le 3$.

Jan. 2007 ($t = 3$): $n(3) = 1.5(3)^2 - 3 + 1.5 = 12$ Use the first formula because $0 \le t \le 3$.

June 2008 ($t = 4.5$): $n(4.5) = 46(4.5) - 126 = 81$ Use the second formula because $3 < t \le 5$.

Thus, the number of Facebook members was approximately 2 million in January 2005, 12 million in January 2007, and 81 million in June 2008.

To sketch the graph of n, we use a table of values of $n(t)$ (some of which we have already calculated above), plot the points, and connect them to sketch the graph:

t	0	1	2	3	3.5	4	4.5	5
$n(t)$	1.5	2	5.5	12	35	58	81	104

$\underbrace{\qquad\qquad}_{\text{First formula}}$ $\underbrace{\qquad\qquad}_{\text{Second formula}}$

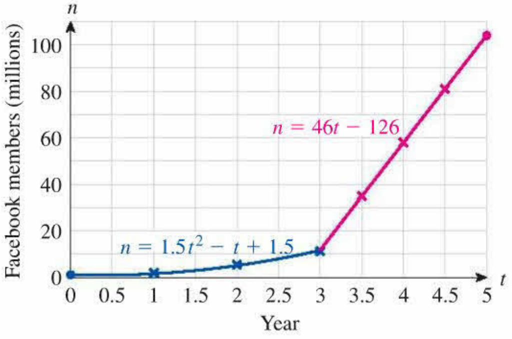


Figure 5

The graph (Figure 5) has the following features:

1. The first formula (the curve) is used for $0 \le t \le 3$.

2. The second formula (ascending line) is used for $3 < t \le 5$.

using Technology

See the Technology Guides at the end of the chapter for detailed instructions on how to obtain the values and graph of n in Example 2 using a TI-83/84 Plus or Excel. Here is an outline:

TI-83/84 Plus

Table of values:
`Y1=(X≤3)(1.5X^2-X+1.5)+(X>3)(46X-126)`
[2ND] [TABLE].
Graph: [WINDOW]; Xmin = 0, Xmax = 5; [ZOOM] [0].
[More details on page 110.]

Excel

Table of values: Headings t and $n(t)$ in A1 and B1; t-values 0, 0.5, 1, 1.5, . . . , 5 in A2–A12.
`=(A2<=3)*(1.5*A2^2-A2+1.5)+(A2>3)*(46*A2-126)`
in B2; copy down through B12.
Graph: Highlight A1–B10; insert Scatter chart. [More details on page 117.]

Web Site

www.FiniteandCalc.org
Go to the Function Evaluator and Grapher under Online Utilities, and enter
`(x<=3)(1.5x^2-x+1.5)+(x>3)(46x-126)`
for y_1. To obtain a table of values, enter the x-values 0, 0.5, 1, 1.5, . . . , 5 in the Evaluator box, and press "Evaluate."
Graph: Set Xmin = 0 and Xmax = 5, and press "Plot Graphs."

---

* Source for data: www.facebook.com/press.php.

3. The domain is [0, 5], so the graph is cut off at $t = 0$ and $t = 5$.

4. The solid dots at the ends indicate the endpoints of the domain.

using Technology

See the Technology Guides at the end of the chapter for detailed instructions on how to obtain the values and graph of *f* in Example 3 using a TI-83/84 Plus or Excel. Here is an outline:

TI-83/84 Plus

```
Y1=(X<-1)*(-1)+
(-1≤X)*(X≤1)*X+
(1<X)*(X²-1)
```

Table of values: `2ND` `TABLE`.

Graph: `WINDOW`; Xmin = −4, Xmax = 2; `ZOOM` `0`.

[More details on page 111.]

Excel

Headings *x* and *f*(*x*) in A1 and B1; *x*-values −4, −3.9, . . . , 2 in A2–A62.

```
=(A2<-1)*(-1)+
(-1<=A2)*(A2<=1)*A2
+(1<A2)*(A2^2-1)
```

in B2; copy down through B62.

Graph: Highlight A1–B62; insert Scatter chart. [More details on page 117.]

Web Site

www.FiniteandCalc.org

Go to the Function Evaluator and Grapher under Online Utilities and enter

```
=(x<-1)*(-1)+(-1<=x)*
(x<=1)*x+(1<x)*(x^2-1)
```

for y_1. To obtain a table of values, enter the *x*-values 0, 0.5, 1, 1.5, . . . , 3 in the Evaluator box and press “Evaluate.”

Graph: Set Xmin = −4 and Xmax = 2, and press “Plot Graphs.”

EXAMPLE 3 More Complicated Piecewise-Defined Functions

Let f be the function specified by

$$f(x) = \begin{cases} -1 & \text{if } -4 \le x < -1 \\ x & \text{if } -1 \le x \le 1 \\ x^2 - 1 & \text{if } 1 < x \le 2 \end{cases}.$$

a. What is the domain of f? Find $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$.

b. Sketch the graph of f.

Solution

a. The domain of f is $[-4, 2]$, because $f(x)$ is specified only when $-4 \le x \le 2$.

$f(-2) = -1$ We used the first formula because $-4 \le x < -1$.

$f(-1) = -1$ We used the second formula because $-1 \le x \le 1$.

$f(0) = 0$ We used the second formula because $-1 \le x \le 1$.

$f(1) = 1$ We used the second formula because $-1 \le x \le 1$.

$f(2) = 2^2 - 1 = 3$ We used the third formula because $1 < x \le 2$.

b. To sketch the graph by hand, we first sketch the three graphs $y = -1$, $y = x$, and $y = x^2 - 1$, and then use the appropriate portion of each (Figure 6).

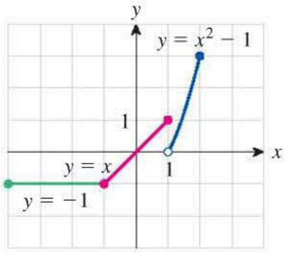


Figure 6

Note that solid dots indicate points on the graph, whereas the open dots indicate points *not* on the graph. For example, when $x = 1$, the inequalities in the formula tell us that we are to use the middle formula (x) rather than the bottom one ($x^2 - 1$). Thus, $f(1) = 1$, not 0, so we place a solid dot at (1, 1) and an open dot at (1, 0).

Vertical Line Test

Every point in the graph of a function has the form $(x, f(x))$ for some x in the domain of f. Because f assigns a *single* value $f(x)$ to each value of x in the domain, it follows that, in the graph of f, there should be only one y corresponding to any such value of x—namely, $y = f(x)$. In other words, *the graph of a function cannot contain two or more points with the same x-coordinate—that is, two or more points on the same vertical line*. On the other hand, a vertical line at a value of x not in the domain will not contain any points in the graph. This gives us the following rule.

Vertical-Line Test

For a graph to be the graph of a function, every vertical line must intersect the graph in *at most* one point.

Quick Examples

As illustrated below, only graph B passes the vertical line test, so only graph B is the graph of a function.

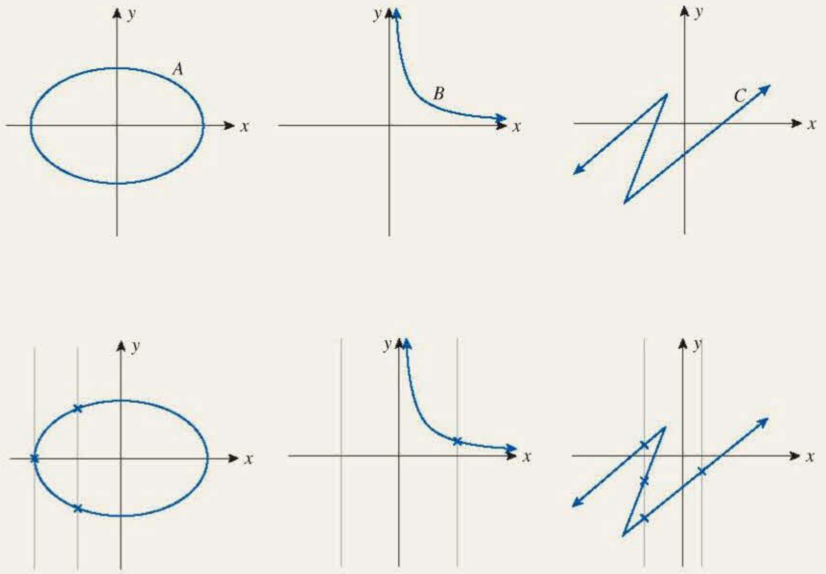


Table 1 lists some common types of functions that are often used to model real world situations.

Table 1 A Compendium of Functions and Their Graphs

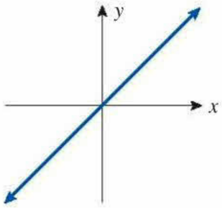
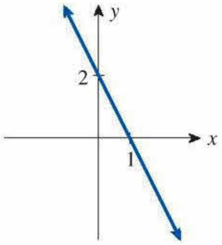
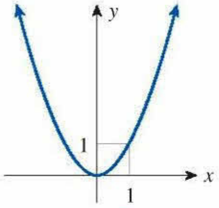
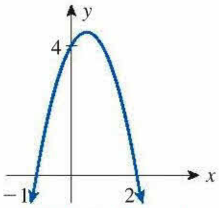
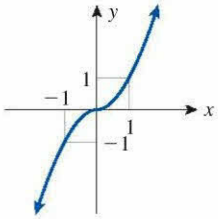
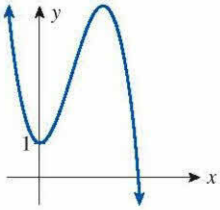
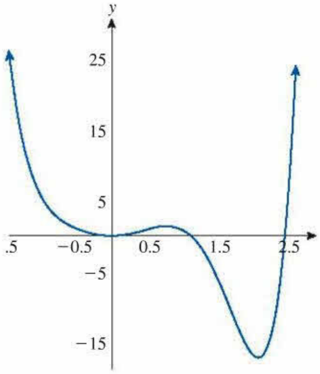
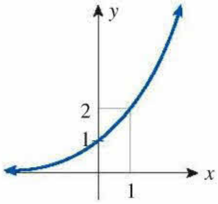
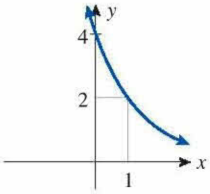
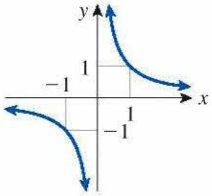
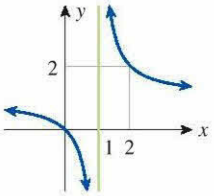
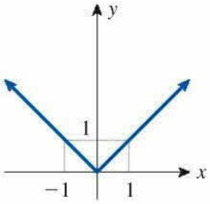
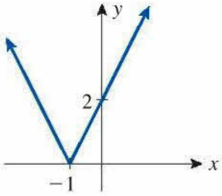
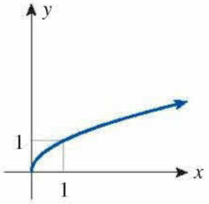
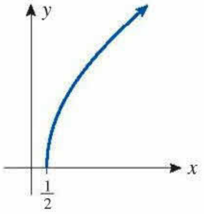
Type of Function	*Examples*	
Linear $f(x) = mx + b$ m, b constant Graphs of linear functions are straight lines. The quantity m is the **slope** of the line; the quantity b is the ***y*-intercept** of the line. [See Section 1.3.]	$y = x$ 	$y = -2x + 2$ 
Technology formulas:	`x`	`-2*x+2`
Quadratic $f(x) = ax^2 + bx + c$ a, b, c constant ($a \neq 0$) Graphs of quadratic functions are called **parabolas**. [See Section 2.1.]	$y = x^2$ 	$y = -2x^2 + 2x + 4$ 
Technology formulas:	`x^2`	`-2*x^2+2*x+4`
Cubic $f(x) = ax^3 + bx^2 + cx + d$ a, b, c, d constant ($a \neq 0$)	$y = x^3$ 	$y = -x^3 + 3x^2 + 1$ 
Technology formulas:	`x^3`	`-x^3+3*x^2+1`
Polynomial $f(x) = ax^n + bx^{n-1} + \ldots + rx + s$ $a, b, \ldots, r, s$ constant (includes all of the above functions)	All the above, and $f(x) = x^6 - 2x^5 - 2x^4 + 4x^2$ 	
Technology formula:	`x^6-2x^5-2x^4+4x^2`	

Table 1 (*Continued*)

Type of Function	*Examples*	
Exponential $f(x) = Ab^x$ A, b constant ($b > 0$ and $b \neq 1$) The y-coordinate is multiplied by b every time x increases by 1.	$y = 2^x$  y is doubled every time x increases by 1.	$y = 4(0.5)^x$  y is halved every time x increases by 1.
Technology formulas:	`2^x`	`4*0.5^x`
Rational $f(x) = \frac{P(x)}{Q(x)}$; $P(x)$ and $Q(x)$ polynomials The graph of $y = 1/x$ is a **hyperbola**. The domain excludes zero because $1/0$ is not defined.	$y = \frac{1}{x}$ 	$y = \frac{x}{x-1}$ 
Technology formulas:	`1/x`	`x/(x-1)`
Absolute value For x positive or zero, the graph of $y = \lvert x\rvert$ is the same as that of $y = x$. For x negative or zero, it is the same as that of $y = -x$.	$y = \lvert x\rvert$ 	$y = \lvert 2x + 2\rvert$ 
Technology formulas:	`abs(x)`	`abs(2*x+2)`
Square Root The domain of $y = \sqrt{x}$ must be restricted to the nonnegative numbers, because the square root of a negative number is not real. Its graph is the top half of a horizontally oriented parabola.	$y = \sqrt{x}$ 	$y = \sqrt{4x - 2}$ 
Technology formulas:	`x^0.5 or √(x)`	`(4*x-2)^0.5 or √(4*x-2)`

Go to the Web site and follow the path

Online Text

→ New Functions from Old: Scaled and Shifted Functions

where you will find complete online interactive text, examples, and exercises on scaling and translating the graph of a function by changing the formula.

Functions and models other than linear ones are called **nonlinear**.

1.1 EXERCISES

▼ more advanced ◆ challenging

T indicates exercises that should be solved using technology

In Exercises 1–4, evaluate or estimate each expression based on the following table. HINT [See Quick Example 1 on page 42.]

x	-3	-2	-1	0	1	2	3
$f(x)$	1	2	4	2	1	0.5	0.25

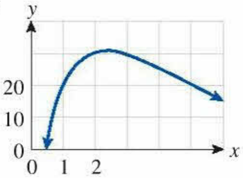
1. a. $f(0)$ **b.** $f(2)$ **2. a.** $f(-1)$ **b.** $f(1)$

3. a. $f(2) - f(-2)$ **b.** $f(-1)f(-2)$ **c.** $-2f(-1)$

4. a. $f(1) - f(-1)$ **b.** $f(1)f(-2)$ **c.** $3f(-2)$

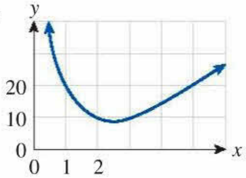
In Exercises 5–8, use the graph of the function f to find approximations of the given values. HINT [See Example 1.]

5. ◆



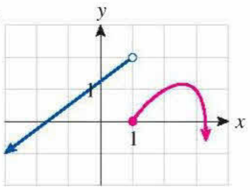
a. $f(1)$ **b.** $f(2)$
c. $f(3)$ **d.** $f(5)$
e. $f(3) - f(2)$

6.



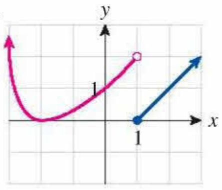
a. $f(1)$ **b.** $f(2)$
c. $f(3)$ **d.** $f(5)$
e. $f(3) - f(2)$

7.



a. $f(-3)$ **b.** $f(0)$
c. $f(1)$ **d.** $f(2)$
e. $\dfrac{f(3) - f(2)}{3 - 2}$

8.



a. $f(-2)$ **b.** $f(0)$
c. $f(1)$ **d.** $f(3)$
e. $\dfrac{f(3) - f(1)}{3 - 1}$

In Exercises 9–12, say whether or not $f(x)$ is defined for the given values of x. If it is defined, give its value. HINT [See Quick Example 3 page 43.]

9. $f(x) = x - \dfrac{1}{x^2}$, with domain $(0, +\infty)$
a. $x = 4$ **b.** $x = 0$ **c.** $x = -1$

10. $f(x) = \dfrac{2}{x} - x^2$, with domain $[2, +\infty)$
a. $x = 4$ **b.** $x = 0$ **c.** $x = 1$

11. $f(x) = \sqrt{x + 10}$, with domain $[-10, 0)$
a. $x = 0$ **b.** $x = 9$ **c.** $x = -10$

12. $f(x) = \sqrt{9 - x^2}$, with domain $(-3, 3)$
a. $x = 0$ **b.** $x = 3$ **c.** $x = -3$

13. Given $f(x) = 4x - 3$, find **a.** $f(-1)$ **b.** $f(0)$
c. $f(1)$ **d.** $f(y)$ **e.** $f(a + b)$ HINT [See Example 1.]

14. Given $f(x) = -3x + 4$, find **a.** $f(-1)$ **b.** $f(0)$
c. $f(1)$ **d.** $f(y)$ **e.** $f(a + b)$

15. Given $f(x) = x^2 + 2x + 3$, find **a.** $f(0)$ **b.** $f(1)$
c. $f(-1)$ **d.** $f(-3)$ **e.** $f(a)$ **f.** $f(x + h)$
HINT [See Example 1.]

16. Given $g(x) = 2x^2 - x + 1$, find **a.** $g(0)$ **b.** $g(-1)$
c. $g(r)$ **d.** $g(x + h)$

17. Given $g(s) = s^2 + \dfrac{1}{s}$, find **a.** $g(1)$ **b.** $g(-1)$
c. $g(4)$ **d.** $g(x)$ **e.** $g(s + h)$ **f.** $g(s + h) - g(s)$

18. Given $h(r) = \dfrac{1}{r + 4}$, find **a.** $h(0)$ **b.** $h(-3)$
c. $h(-5)$ **d.** $h(x^2)$ **e.** $h(x^2 + 1)$ **f.** $h(x^2) + 1$

In Exercises 19–24, graph the given functions. Give the technology formula and use technology to check your graph. We suggest that you become familiar with these graphs in addition to those in Table 2. HINT [See Quick Example 4 on page 44.]

19. $f(x) = -x^3$ (domain $(-\infty, +\infty)$)

20. $f(x) = x^3$ (domain $[0, +\infty)$)

21. $f(x) = x^4$ (domain $(-\infty, +\infty)$)

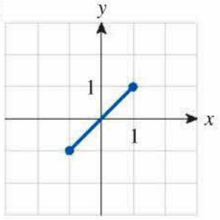
22. $f(x) = \sqrt[3]{x}$ (domain $(-\infty, +\infty)$)

23. $f(x) = \dfrac{1}{x^2}$ $(x \neq 0)$ **24.** $f(x) = x + \dfrac{1}{x}$ $(x \neq 0)$

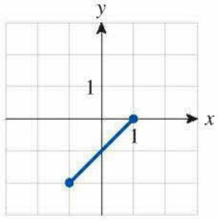
In Exercises 25 and 26, match the functions to the graphs. Using technology to draw the graphs is suggested, but not required.

25. T **a.** $f(x) = x$ $(-1 \le x \le 1)$
b. $f(x) = -x$ $(-1 \le x \le 1)$
c. $f(x) = \sqrt{x}$ $(0 < x < 4)$
d. $f(x) = x + \dfrac{1}{x} - 2$ $(0 < x < 4)$
e. $f(x) = |x|$ $(-1 \le x \le 1)$
f. $f(x) = x - 1$ $(-1 \le x \le 1)$

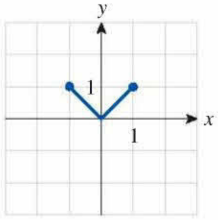
(I)



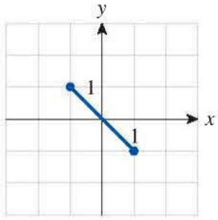
(II)

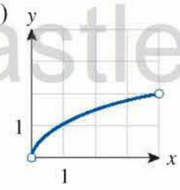


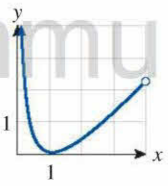
(III)



(IV)



(V)


(VI)


26. T **a.** $f(x) = -x + 4 \quad (0 < x \leq 4)$

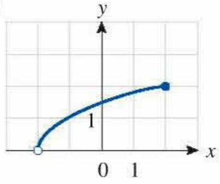
b. $f(x) = 2 - |x| \quad (-2 < x \leq 2)$

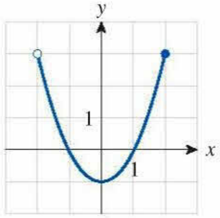
c. $f(x) = \sqrt{x+2} \quad (-2 < x \leq 2)$

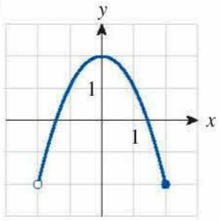
d. $f(x) = -x^2 + 2 \quad (-2 < x \leq 2)$

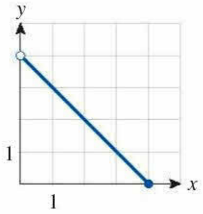
e. $f(x) = \dfrac{1}{x} - 1 \quad (0 < x \leq 4)$

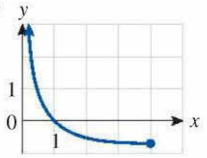
f. $f(x) = x^2 - 1 \quad (-2 < x \leq 2)$

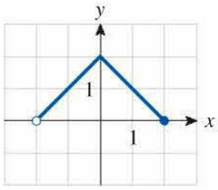
(I)


(II)


(III)


(IV)


(V)


(VI)


T *In Exercises 27–30, first give the technology formula for the given function and then use technology to evaluate the function for the given values of x (when defined there).*

27. T $f(x) = 0.1x^2 - 4x + 5;\ x = 0, 1, \ldots, 10$

28. T $g(x) = 0.4x^2 - 6x - 0.1;\ x = -5, -4, \ldots, 4, 5$

29. T $h(x) = \dfrac{x^2 - 1}{x^2 + 1};\ x = 0.5, 1.5, 2.5, \ldots, 10.5$ (Round all answers to four decimal places.)

30. T $r(x) = \dfrac{2x^2 + 1}{2x^2 - 1};\ x = -1, 0, 1, \ldots, 9$ (Round all answers to four decimal places.)

In Exercises 31–36, sketch the graph of the given function, evaluate the given expressions, and then use technology to duplicate the graphs. Give the technology formula. HINT [See Example 2.]

31. $f(x) = \begin{cases} x & \text{if } -4 \leq x < 0 \\ 2 & \text{if } 0 \leq x \leq 4 \end{cases}$

a. $f(-1)$ **b.** $f(0)$ **c.** $f(1)$

32. $f(x) = \begin{cases} -1 & \text{if } -4 \leq x \leq 0 \\ x & \text{if } 0 < x \leq 4 \end{cases}$

a. $f(-1)$ **b.** $f(0)$ **c.** $f(1)$

33. $f(x) = \begin{cases} x^2 & \text{if } -2 < x \leq 0 \\ 1/x & \text{if } 0 < x \leq 4 \end{cases}$

a. $f(-1)$ **b.** $f(0)$ **c.** $f(1)$

34. $f(x) = \begin{cases} -x^2 & \text{if } -2 < x \leq 0 \\ \sqrt{x} & \text{if } 0 < x < 4 \end{cases}$

a. $f(-1)$ **b.** $f(0)$ **c.** $f(1)$

35. $f(x) = \begin{cases} x & \text{if } -1 < x \leq 0 \\ x+1 & \text{if } 0 < x \leq 2 \\ x & \text{if } 2 < x \leq 4 \end{cases}$

a. $f(0)$ **b.** $f(1)$ **c.** $f(2)$ **d.** $f(3)$ HINT [See Example 3.]

36. $f(x) = \begin{cases} -x & \text{if} -1 < x \leq 0 \\ x-2 & \text{if } 0 \leq x \leq 2 \\ -x & \text{if } 2 < x \leq 4 \end{cases}$

a. $f(0)$ **b.** $f(1)$ **c.** $f(2)$ **d.** $f(3)$

In Exercises 37–40, find and simplify ***(a)*** $f(x+h) - f(x)$ ***(b)*** $\dfrac{f(x+h) - f(x)}{h}$

37. ▼ $f(x) = x^2$

38. ▼ $f(x) = 3x - 1$

39. ▼ $f(x) = 2 - x^2$

40. ▼ $f(x) = x^2 + x$

APPLICATIONS

41. ***Oil Imports from Mexico*** The following table shows U.S. oil imports from Mexico, for 2001–2006 ($t = 1$ represents 2001):[2]

***t* (year since 2000)**	1	2	3	4	5	6
***I* (million gallons/day)**	1.35	1.5	1.55	1.6	1.5	1.5

a. Find $I(3)$, $I(5)$, and $I(6)$. Interpret your answers.

b. What is the domain of I?

c. Represent I graphically, and use your graph to estimate $I(4.5)$. Interpret your answer. HINT [See Quick Example 1 on page 42.]

---

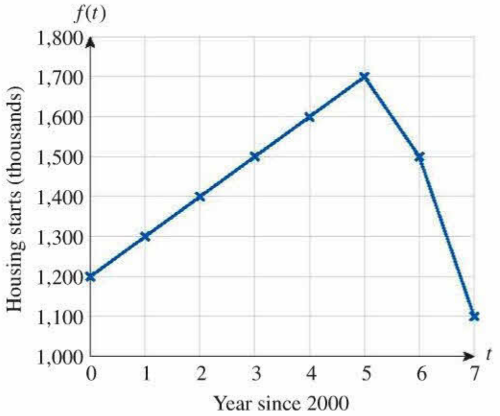
[2] Figures are approximate. Source: Energy Information Administration: Pemex/*New York Times*, March 9, 2007, p. C4.

42. ***Oil Production in Mexico*** The following table shows oil production by Pemex, Mexico's national oil company, for 2001–2006 ($t = 1$ represents 2001):[3]

***t* (year since 2000)**	1	2	3	4	5	6
***P* (million gallons/day)**	3.1	3.2	3.4	3.4	3.4	3.3

a. Find $P(3)$, $P(4)$, and $P(6)$. Interpret your answers.
b. What is the domain of P?
c. Represent P graphically, and use your graph to estimate $P(1.5)$. Interpret your answer.

Housing Starts *Exercises 43–46 refer to the following graph, which shows the number f(t) of housing starts in the U.S. each year from 2000 through 2007 (t = 0 represents 2000, and f(t) is the number of housing starts in year t in thousands of units).*[4]



43. Estimate $f(4)$, $f(5)$, and $f(6.5)$. Interpret your answers.

44. Estimate $f(3)$, $f(6)$, and $f(5.5)$. Interpret your answers.

45. Which has the larger magnitude: $f(5) - f(0)$ or $f(7) - f(5)$? Interpret the answer.

46. Which has the larger magnitude: $f(7) - f(6)$ or $f(5) - f(0)$? Interpret the answer.

47. ***Trade with China*** The value of U.S. trade with China from 1994 through 2004 can be approximated by

$$C(t) = 3t^2 - 7t + 50 \text{ billion dollars}$$

(t is time in years since 1994).[5]

a. Find an appropriate domain of C. Is $t \geq 0$ an appropriate domain? Why or why not?
b. Compute $C(10)$. What does the answer say about trade with China?

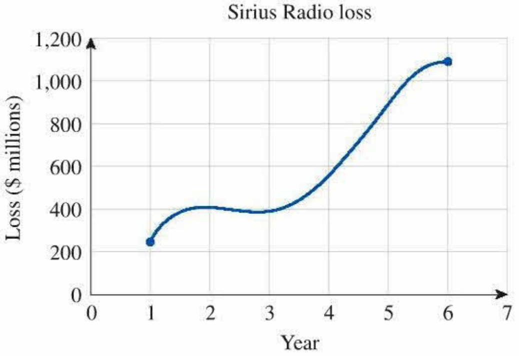
48. ***Scientific Research*** The number of research articles in *Physical Review* that were written by researchers in the United States from 1983 through 2003 can be approximated by

$$A(t) = -0.01t^2 + 0.24t + 3.4 \text{ hundred articles}$$

(t is time in years since 1983).[6]

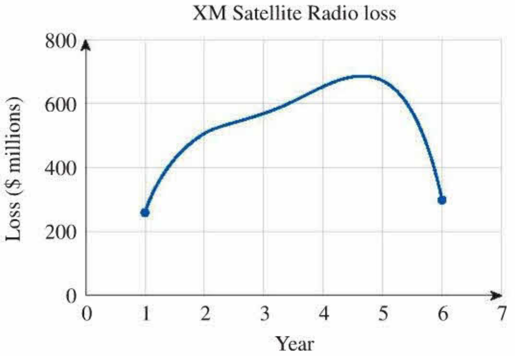
a. Find an appropriate domain of A. Is $t \leq 20$ an appropriate domain? Why or why not?
b. Compute $A(10)$. What does the answer say about the number of research articles?

49. ***Satellite Radio Losses*** The following graph shows the approximate annual loss $L(t)$ of Sirius Satellite Radio for the perod 2001–2006 ($t = 1$ represents the start of 2001).[7]



a. What is the domain of L?
b. Estimate $L(2)$, $L(5)$, and $L(6)$. Interpret your answers.
c. At approximately which value of t is $L(t)$ increasing most rapidly? Interpret the result.

50. ***Satellite Radio Losses*** The following graph shows the approximate annual loss $L(t)$ of XM Satellite Radio for the period 2001–2006 ($t = 1$ represents the start of 2001).



a. What is the domain of L?
b. Estimate $L(2)$, $L(3.5)$, and $L(6)$. Interpret your answers.
c. At approximately which value of t is $L(t)$ increasing most rapidly? Interpret the result.

---

[3] Figures are approximate. Source: Energy Information Administration: Pemex/*New York Times*, March 9, 2007, p. C4.

[4] Figures are rounded. Source: www.census.gov (2007).

[5] Based on a regression by the authors. Source for data: U.S. Census Bureau/*New York Times*, September 23, 2004, p. C1.

[6] Based on a regression by the authors. Source for data: The American Physical Society/*New York Times*, May 3, 2003, p. A1.

[7] Source: Bloomberg Financial Markets, Sirius Satellite Radio/*New York Times*, February 20, 2008, p. C2.

51. ***Processor Speeds*** The processor speed, in megahertz, of Intel processors could be approximated by the following function of time t in years since the start of 1995:[8]

$$P(t) = \begin{cases} 75t + 200 & \text{if } 0 \le t \le 4 \\ 600t - 1900 & \text{if } 4 < t \le 9 \end{cases}$$

a. Evaluate $P(0)$, $P(4)$, and $P(5)$ and interpret the results.
b. Sketch the graph of P and use your graph to estimate when processor speeds first reached 2.0 gigahertz (1 gigahertz = 1,000 megahertz).
c. T Use technology to generate a table of values for $P(t)$ with $t = 0, 1, \ldots, 9$.

52. ***Leading Economic Indicators*** The value of the Conference Board Index of 10 economic indicators in the United States could be approximated by the following function of time t in months since the end of December 2002:[9]

$$E(t) = \begin{cases} 0.4t + 110 & \text{if } 6 \le t \le 15 \\ -0.2t + 119 & \text{if } 15 < t \le 20 \end{cases}$$

a. Estimate $E(10)$, $E(15)$, and $E(20)$ and interpret the results.
b. Sketch the graph of E and use your graph to estimate when the index first reached 115.
c. T Use technology to generate a table of values for $E(t)$ with $t = 6, 7, \ldots, 20$.

53. ▼ ***Income Taxes*** The U.S. Federal income tax is a function of taxable income. Write T for the tax owed on a taxable income of I dollars. For tax year 2007, the function T for a single taxpayer was specified as follows:

If your taxable income was			*of the*
Over—	*But not over—*	*Your tax is*	*amount over—*
\$0	7,825	.... 10%	**\$0**
7,825	31,850	**\$782.50 + 15%**	**\$7,825**
31,850	77,100	**4,386.25 + 25%**	**\$31,850**
77,100	160,850	**15,698.75 + 28%**	**\$77,100**
160,850	349,700	**39,148.75 + 33%**	**\$160,850**
349,700	....	**101,469.25 + 35%**	**\$349,700**

What was the tax owed by a single taxpayer on a taxable income of \$26,000? On a taxable income of \$65,000?

54. ▼ ***Income Taxes*** The income tax function T in Exercise 53 can also be written in the following form:

$$T(I) = \begin{cases} 0.10I & \text{if } 0 < I \le 7{,}825 \\ 782.50 + 0.15(I - 7{,}825) & \text{if } 7{,}825 < I \le 31{,}850 \\ 4{,}386.25 + 0.25(I - 31{,}850) & \text{if } 31{,}850 < I \le 77{,}100 \\ 15{,}698.75 + 0.28(I - 77{,}100) & \text{if } 77{,}100 < I \le 160{,}850 \\ 39{,}148.75 + 0.33(I - 160{,}850) & \text{if } 160{,}850 < I \le 349{,}700 \\ 101{,}469.25 + 0.35(I - 349{,}700) & \text{if } I > 349{,}700 \end{cases}$$

What was the tax owed by a single taxpayer on a taxable income of \$25,000? On a taxable income of \$125,000?

55. T ▼ ***Acquisition of Language*** The percentage $p(t)$ of children who can speak in at least single words by the age of t months can be approximated by the equation[10]

$$p(t) = 100\left(1 - \frac{12{,}200}{t^{4.48}}\right) \quad (t \ge 8.5)$$

a. Give a technology formula for p.
b. Graph p for $8.5 \le t \le 20$ and $0 \le p \le 100$.
c. Create a table of values of p for $t = 9, 10, \ldots, 20$ (rounding answers to one decimal place).
d. What percentage of children can speak in at least single words by the age of 12 months?
e. By what age are 90% or more children speaking in at least single words?

56. T ▼ ***Acquisition of Language*** The percentage $p(t)$ of children who can speak in sentences of five or more words by the age of t months can be approximated by the equation[11]

$$p(t) = 100\left(1 - \frac{5.27 \times 10^{17}}{t^{12}}\right) \quad (t \ge 30)$$

a. Give a technology formula for p.
b. Graph p for $30 \le t \le 45$ and $0 \le p \le 100$.
c. Create a table of values of p for $t = 30, 31, \ldots, 40$ (rounding answers to one decimal place).
d. What percentage of children can speak in sentences of five or more words by the age of 36 months?
e. By what age are 75% or more children speaking in sentences of five or more words?

COMMUNICATION AND REASONING EXERCISES

57. If the market price m of gold varies with time t, then the independent variable is ___ and the dependent variable is ___.

58. Complete the following sentence: If weekly profit P is specified as a function of selling price s, then the independent variable is ___ and the dependent variable is ___.

59. Complete the following: The function notation for the equation $y = 4x^2 - 2$ is ____.

---

[8] Source: Sandpile.org/*New York Times*, May 17, 2004, p. C1.

[9] Source: The Conference Board/*New York Times*, November 19, 2004, p. C7.

[10] The model is the authors' and is based on data presented in the article *The Emergence of Intelligence* by William H. Calvin, *Scientific American*, October, 1994, pp. 101–107.

[11] Ibid.

60. Complete the following: The equation notation for $C(t) = -0.34t^2 + 0.1t$ is ____.

61. True or false? Every graphically specified function can also be specified numerically. Explain.

62. True or false? Every algebraically specified function can also be specified graphically. Explain.

63. True or false? Every numerically specified function with domain [0, 10] can also be specified algebraically. Explain.

64. True or false? Every graphically specified function can also be specified algebraically. Explain.

65. ▼ True or false? Every function can be specified numerically.

66. ▼ Which supplies more information about a situation: a numerical model or an algebraic model?

67. ▼ Why is the following assertion false? "If $f(x) = x^2 - 1$, then $f(x + h) = x^2 + h - 1$."

68. ▼ Why is the following assertion false? "If $f(2) = 2$ and $f(4) = 4$, then $f(3) = 3$."

69. How do the graphs of two functions differ if they are specified by the same formula but have different domains?

70. How do the graphs of two functions $f(x)$ and $g(x)$ differ if $g(x) = f(x) + 10$? (Try an example.)

71. ▼ How do the graphs of two functions $f(x)$ and $g(x)$ differ if $g(x) = f(x - 5)$? (Try an example.)

72. ▼ How do the graphs of two functions $f(x)$ and $g(x)$ differ if $g(x) = f(-x)$? (Try an example.)

1.2 Functions and Models

The functions we used in Examples 1 and 2 in Section 1.1 are **mathematical models** of real-life situations, because they model, or represent, situations in mathematical terms.

Mathematical Modeling

To mathematically model a situation means to represent it in mathematical terms. The particular representation used is called a **mathematical model** of the situation. Mathematical models do not always represent a situation perfectly or completely. Some (like Example 1 of Section 1.1) represent a situation only approximately, whereas others represent only some aspects of the situation.

Quick Examples

Situation	Model
1. The temperature is now 10°F and increasing by 20° per hour.	$T(t) = 10 + 20t$ (t = time in hours, T = temperature)
2. I invest $1,000 at 5% interest compounded quarterly. Find the value of the investment after t years.	$A(t) = 1{,}000\left(1 + \frac{0.05}{4}\right)^{4t}$ This is the compound interest formula we will study in Example 6.
3. I am fencing a rectangular area whose perimeter is 100 ft. Find the area as a function of the width x.	Take y to be the length, so the perimeter is $100 = x + y + x + y = 2(x + y)$ so $x + y = 50$. Thus the length is $y = 50 - x$. Area $A = xy = x(50 - x)$.
4. iPod sales	The function $f(x) = -x^2 + 20x + 3$ in Example 1 of Section 1.1 is an **algebraic model** of iPod sales.

5. Facebook membership

The function

$$n(t) = \begin{cases} 1.5t^2 - t + 1.5 & \text{if } 0 \le t \le 3 \\ 46t - 126 & \text{if } 3 < t \le 4 \end{cases}$$

in Example 2 of Section 1.1 is a **piecewise algebraic model** of Facebook membership.

Analytical and Curve-Fitting Models

Quick Examples 1–3 are **analytical models**, obtained by analyzing the situation being modeled, whereas Quick Examples 4 and 5 are **curve-fitting models**, obtained by finding mathematical formulas that approximate observed data.

Cost, Revenue, and Profit Models

Photodisc/Getty Images

EXAMPLE 1 Modeling Cost: Cost Function

As of October 2007, Yellow Cab Chicago's rates were \$1.90 on entering the cab plus \$1.60 for each mile.*

a. Find the cost C of an x-mile trip.

b. Use your answer to calculate the cost of a 40-mile trip.

c. What is the cost of the second mile? What is the cost of the tenth mile?

d. Graph C as a function of x.

Solution

a. We are being asked to find how the cost C depends on the length x of the trip, or to find C as a function of x. Here is the cost in a few cases:

Cost of a 1-mile trip: $C = 1.60(1) + 1.90 = 3.50$ 1 mile @ \$1.60 per mile plus \$1.90

Cost of a 2-mile trip: $C = 1.60(2) + 1.90 = 5.10$ 2 miles @ \$1.60 per mile plus \$1.90

Cost of a 3-mile trip: $C = 1.60(3) + 1.90 = 6.70$ 3 miles @ \$1.60 per mile plus \$1.90

Do you see the pattern? The cost of an x-mile trip is given by the linear function

$$C(x) = 1.60x + 1.90.$$

Notice that the cost function is a sum of two terms: the **variable cost** $1.60x$, which depends on x, and the **fixed cost** 1.90, which is independent of x:

$$\text{Cost} = \text{Variable Cost} + \text{Fixed Cost}.$$

The quantity 1.60 by itself is the incremental cost per mile; you might recognize it as the *slope* of the given linear function. In this context we call 1.60 the **marginal cost**. You might recognize the fixed cost 1.90 as the *C-intercept* of the given linear function.

---

* According to their Web site at www.yellowcabchicago.com/.

b. We can use the formula for the cost function to calculate the cost of a 40-mile trip as

$$C(40) = 1.60(40) + 1.90 = \$65.90.$$

c. To calculate the cost of the second mile, we *could* proceed as follows:

Find the cost of a 1-mile trip: $C(1) = 1.60(1) + 1.90 = \$3.50$.

Find the cost of a 2-mile trip: $C(2) = 1.60(2) + 1.90 = \$5.10$.

Therefore, the cost of the second mile is $\$5.10 - \$3.50 = \$1.60$.

But notice that this is just the marginal cost. In fact, the marginal cost is the cost of each additional mile, so we could have done this more simply:

Cost of second mile = Cost of tenth mile = Marginal cost = \$1.60

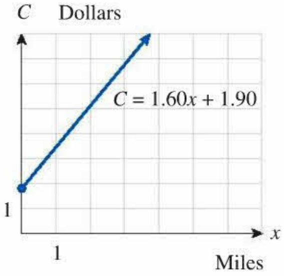


Figure 7

d. Figure 7 shows the graph of the cost function, which we can interpret as a *cost vs. miles* graph. The fixed cost is the starting height on the left, while the marginal cost is the slope of the line: It rises 1.60 units per unit of x. (See Section 1.3 for a discussion of properties of straight lines.)

➡ **Before we go on...** The cost function in Example 1 is an example of an *analytical model:* We derived the form of the cost function from a knowledge of the cost per mile and the fixed cost.

As we discussed on page 46 in Section 1.1, we can use equation notation to specify a function. In equation notation, the function C in Example 1 is

$$C = 1.60x + 1.90.$$ Equation notation

The independent variable is x and the dependent variable is C. Function notation and equation notation, using the same letter for the function name and the dependent variable, are often used interchangeably. It is important to be able to switch back and forth between function notation and equation notation easily. ■

Here is a summary of some terms we used in Example 1, along with an introduction to some new terms:

Cost, Revenue, and Profit Functions

A **cost function** specifies the cost C as a function of the number of items x. Thus, $C(x)$ is the cost of x items, and has the form

Cost = Variable cost + Fixed cost

where the variable cost is a function of x and the fixed cost is a constant. A cost function of the form

$$C(x) = mx + b$$

is called a **linear cost function**; the variable cost is mx and the fixed cost is b. The slope m, the **marginal cost**, measures the incremental cost per item.

The **revenue** resulting from one or more business transactions is the total payment received, sometimes called the gross proceeds. If $R(x)$ is the revenue from selling x items at a price of m each, then R is the linear function $R(x) = mx$ and the selling price m can also be called the **marginal revenue**.

The **profit**, on the other hand, is the *net* proceeds, or what remains of the revenue when costs are subtracted. If the profit depends linearly on the number of items, the slope m is called the **marginal profit**. Profit, revenue, and cost are related by the following formula.

$$\text{Profit} = \text{Revenue} - \text{Cost}$$
$$P = R - C$$

If the profit is negative, say $-\$500$, we refer to a **loss** (of \$500 in this case). To **break even** means to make neither a profit nor a loss. Thus, break even occurs when $P = 0$, or

$$R = C. \qquad \text{Break even}$$

The **break-even point** is the number of items x at which break even occurs.

Quick Example

If the daily cost (including operating costs) of manufacturing x T-shirts is $C(x) = 8x + 100$, and the revenue obtained by selling x T-shirts is $R(x) = 10x$, then the daily profit resulting from the manufacture and sale of x T-shirts is

$$P(x) = R(x) - C(x) = 10x - (8x + 100) = 2x - 100.$$

Break even occurs when $P(x) = 0$, or $x = 50$.

EXAMPLE 2 Cost, Revenue, and Profit

The annual operating cost of *YSport Fitness* gym is estimated to be

$$C(x) = 100{,}000 + 160x - 0.2x^2 \qquad (0 \le x \le 400)$$

where x is the number of members. Annual revenue from membership averages \$800 per member. What is the variable cost? What is the fixed cost? What is the profit function? How many members must *YSport* have to make a profit? What will happen if it has fewer members? If it has more?

Solution The variable cost is the part of the cost function that depends on x:

$$\text{Variable cost} = 160x - 0.2x^2.$$

The fixed cost is the constant term:

$$\text{Fixed cost} = 100{,}000.$$

The annual revenue *YSport* obtains from a single member is \$800. So, if it has x members, it earns an annual revenue of

$$R(x) = 800x.$$

For the profit, we use the formula

$$\begin{aligned} P(x) &= R(x) - C(x) && \text{Formula for profit} \\ &= 800x - (100{,}000 + 160x - 0.2x^2) && \text{Substitute } R(x) \text{ and } C(x) \\ &= -100{,}000 + 640x + 0.2x^2. \end{aligned}$$

using Technology

Excel has a feature called "Goal Seek" which can be used to find the point of intersection of the cost and revenue graphs numerically rather than graphically. See the downloadable Excel tutorial for this section at the Web site.

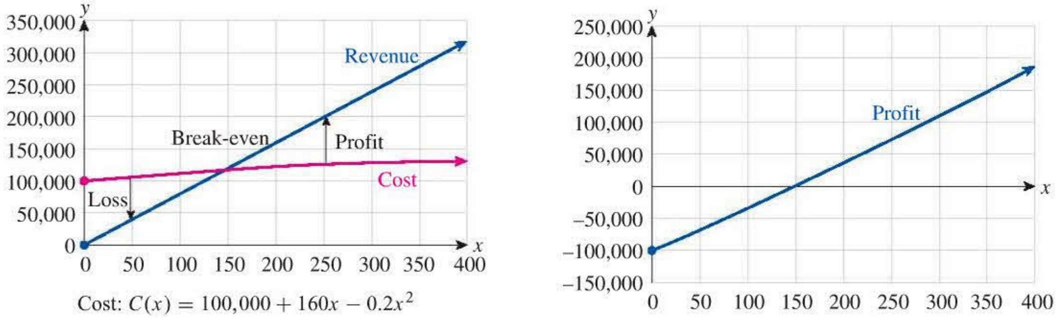
To make a profit, *YSport* needs to do better than break even, so let us find the break-even point: the value of x such that $P(x) = 0$. All we have to do is set $P(x) = 0$ and solve for x:

$$-100{,}000 + 640x + 0.2x^2 = 0.$$

Notice that we have a quadratic equation $ax^2 + bx + c = 0$ with $a = 0.2$, $b = 640$, and $c = -100{,}000$. Its solution is given by the quadratic formula:

$$\begin{aligned} x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-640 \pm \sqrt{640^2 + 4(0.2)(100{,}000)}}{2(0.2)} \\ &\approx \frac{-640 \pm 699.71}{2(0.2)} \\ &\approx 149.3 \text{ or } -3{,}349.3. \end{aligned}$$

We reject the negative solution (as the domain is [0, 400]) and conclude that $x \approx 149.3$ members. To make a profit, should *YSport* have 149 members or 150 members? To decide, take a look at Figure 8, which shows two graphs: On the top we see the graph of revenue and cost, and on the bottom we see the graph of the profit function.



Cost: $C(x) = 100{,}000 + 160x - 0.2x^2$
Revenue: $R(x) = 800x$
Break even occurs at the point of intersection of the graphs of revenue and cost.

Profit: $P(x) = -100{,}000 + 640x + 0.2x^2$
Break even occurs when $P(x) = 0$

Figure 8

For values of x less than the break-even point of 149.3, $P(x)$ is negative, so the company will have a loss. For values of x greater than the break-even point, $P(x)$ is positive, so the company will make a profit. Thus, *YSport Fitness* needs at least 150 members to make a profit. (Note that we rounded 149.3 up to 150 in this case.)

Demand and Supply Models

The demand for a commodity usually goes down as its price goes up. It is traditional to use the letter q for the (quantity of) demand, as measured, for example, in sales. Consider the following example.

EXAMPLE 3 Demand: Private Schools

The demand for private schools in Michigan depends on the tuition cost and can be approximated by

$$q = 77.8p^{-0.11} \text{ thousand students} \qquad (200 \le p \le 2{,}200) \qquad \text{Demand curve}$$

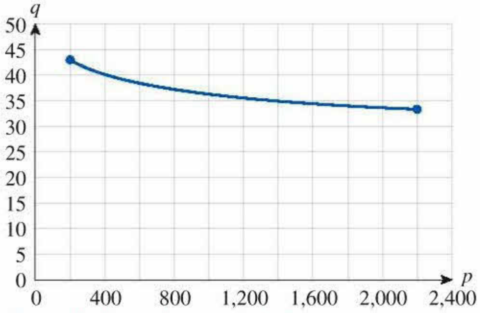
where p is the net tuition cost in dollars.*

a. Use technology to plot the demand function.

b. What is the effect on demand if the tuition cost is increased from \$1,000 to \$1,500?

Solution

a. The demand function is given by $q(p) = 77.8p^{-0.11}$. Its graph is known as a **demand curve**. (Figure 9)



Technology Formula: `y = 77.8*x^(-0.11)`

Figure 9

b. The demand at tuition costs of \$1,000 and \$1,500 are

$$q(1{,}000) = 77.8(1{,}000)^{-0.11} \approx 36.4 \text{ thousand students}$$
$$q(1{,}500) = 77.8(1{,}500)^{-0.11} \approx 34.8 \text{ thousand students}$$

The change in demand is therefore

$$q(1{,}500) - q(1{,}000) \approx 34.8 - 36.4 = -1.6 \text{ thousand students}$$

---

* The tuition cost is net cost: tuition minus tax credit. The model is based on data in "The Universal Tuition Tax Credit: A Proposal to Advance Personal Choice in Education," Patrick L. Anderson, Richard McLellan, J.D., Joseph P. Overton, J.D., Gary Wolfram, Ph.D., Mackinac Center for Public Policy, www.mackinac.org/

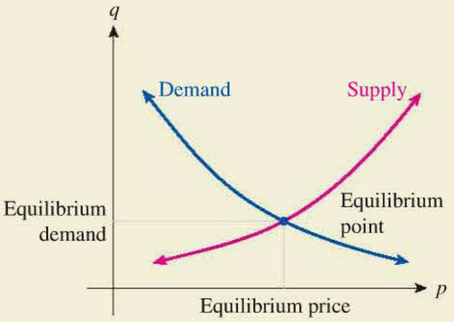
We have seen that a demand function gives the number of items consumers are willing to buy at a given price, and a higher price generally results in a lower demand. However, as the price rises, suppliers will be more inclined to produce these items (as opposed to spending their time and money on other products), so supply will generally rise. A **supply function** gives q, the number of items suppliers are willing to make available for sale,* as a function of p, the price per item.

* **NOTE** Although a bit confusing at first, it is traditional to use the same letter q for the quantity of supply and the quantity of demand, particularly when we want to compare them, as in the next example.

Demand, Supply, and Equilibrium Price

A **demand equation** or **demand function** expresses demand q (the number of items demanded) as a function of the unit price p (the price per item). A **supply equation** or **supply function** expresses supply q (the number of items a supplier is willing to make available) as a function of the unit price p (the price per item). It is usually the case that demand decreases and supply increases as the unit price increases.

Demand and supply are said to be in **equilibrium** when demand equals supply. The corresponding values of p and q are called the **equilibrium price** and **equilibrium demand**. To find the equilibrium price, determine the unit price p where the demand and supply curves cross (sometimes we can determine this value analytically by setting demand equal to supply and solving for p). To find the equilibrium demand, evaluate the demand (or supply) function at the equilibrium price.



Quick Example

If the demand for your exclusive T-shirts is $q = -20p + 800$ shirts sold per day and the supply is $q = 10p - 100$ shirts per day, then the equilibrium point is obtained when demand = supply:

$$-20p + 800 = 10p - 100$$

$$30p = 900, \text{ giving } p = \$30$$

The equilibrium price is therefore \$30 and the equilibrium demand is $q = -20(30) + 800 = 200$ shirts per day. What happens at prices other than the equilibrium price is discussed in Example 4.

EXAMPLE 4 Demand, Supply, and Equilibrium Price

Continuing with Example 3, suppose that private school institutions are willing to create private schools to accommodate

$$q = 30.4 + 0.006p \text{ thousand students} \qquad (200 \le p \le 2{,}200) \qquad \text{Supply curve}$$

who pay a net tuition of p dollars.

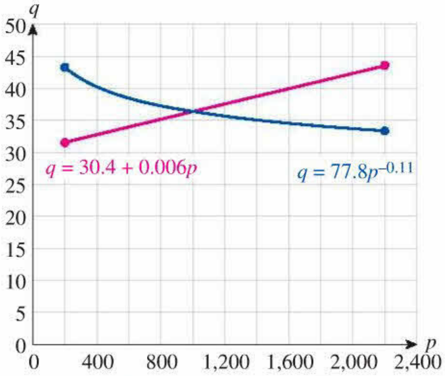
a. Graph the demand curve of Example 3 and the supply curve given here on the same set of axes. Use your graph to estimate, to the nearest \$100, the tuition at which the demand equals the supply. Approximately how many students will be accommodated at that price, known as the **equilibrium price**?

b. What happens if the tuition is higher than the equilibrium price? What happens if it is lower?

c. Estimate the shortage or surplus of openings at private schools if tuition is set at \$1,200.

Solution

a. Figure 10 shows the graphs of demand $q = 77.8p^{-0.11}$ and supply $q = 30.4 + 0.006p$. (See the margin note for a brief description of how to plot them.)



Demand: $q = 77.8p^{-0.11}$
Supply: $q = 30.4 + 0.006p$

Figure 10

The lines cross close to $p = \$1,000$, so we conclude that demand = supply when $p \approx \$1,000$ (to the nearest \$100). This is the (approximate) equilibrium tuition price. At that price, we can estimate the demand or supply at around

Demand: $q = 77.8(1000)^{-0.11} \approx 36.4$

Supply: $q = 30.4 + 0.006(1000) = 36.4$ Demand = Supply at equilibrium

or 36,400 students.

b. Take a look at Figure 11, which shows what happens if schools charge more or less than the equilibrium price. If tuition is, say, \$1,800, then the supply will be larger than demand and there will be a surplus of available openings at private schools. Similarly, if tuition is less—say \$400—then the supply will be less than the demand, and there will be a shortage of available openings.

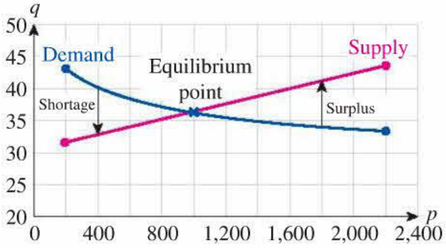


Figure 11

c. The discussion in part (b) shows that if tuition is set at \$1,200 there will be a surplus of available openings. To estimate that number, we calculate the projected demand and supply when $p = \$1,200$:

Demand: $q = 77.8(1,200)^{-0.11} \approx 35.7$ thousand seats

Supply: $q = 30.4 + 0.006(1,200) = 37.6$ thousand seats

Surplus = Supply − Demand $\approx 37.6 - 35.7 = 1.9$ thousand seats

So, there would be a surplus of around 1,900 available seats.

using Technology

See the Technology Guides at the end of the chapter for detailed instructions on how to obtain the graph in Example 4 using a TI-83/84 Plus or Excel. Here is an outline:

TI-83/84 Plus

Graphs:
Y_1`=77.8*X^(-0.11)`
Y_2`=30.4+0.006*X`
Xmin = 200, Xmax = 2200;
`ZOOM` `0` . [More details on page 111.]

Excel

Graphs: Headings *p*, Demand, Supply in A1–C1; *p*-values 200, 300, . . . , 2200 in A2–A22.
`=77.8*A2^(-0.11)` in B2
`=30.4+0.006*A2` in C2
copy down through C22.
Highlight A1–C22; insert Scatter chart. [More details on page 118.]

Web Site

www.FiniteandCalc.org
Go to the Function Evaluator and Grapher under Online Utilities, enter
`77.8*x^(-0.11)` for y_1 and `30.4+0.006*x` for y_2.
Graph: Set Xmin = 200 and Xmax = 2200, and press "Plot Graphs."

➡ **Before we go on...** We just saw in Example 4 that if tuition is less than the equilibrium price there will be a shortage. If schools were to raise their tuition toward the equilibrium, they could create and fill more openings and increase revenue, because it is the supply equation—and not the demand equation—that determines what one can sell below the equilibrium price. On the other hand, if they were to charge more than the equilibrium price, they will be left with a possibly costly surplus of unused openings (and will want to lower tuition to reduce the surplus). Prices tend to move toward the equilibrium, so supply tends to equal demand. When supply equals demand, we say that the market **clears**. ■

Modeling Change Over Time

Things around us change with time. Thus, there are many quantities, such as your income or the temperature in Honolulu, that are natural to think of as functions of time. Example 1 on page 44 (on iPod sales) and Example 2 on page 47 (on Facebook membership) in Section 1.1 are models of change over time. Both of those models are curve-fitting models: We used algebraic functions to approximate observed data.

In the next example we are asked to select from among several curve-fitting models for given data.

EXAMPLE 5 Model Selection: Casino Revenue

The following table shows annual revenues in 2000–2006 from slot machines and video poker machines at the *Mohegan Sun* casino in Connecticut ($t = 0$ represents 2000).*

t	0	1	2	3	4	5	6
Revenue ($ million)	550	680	750	820	850	890	910

Consider the following four models:

(1) $r(t) = 57t + 607$ Linear model

(2) $r(t) = -9t^2 + 110t + 560$ Quadratic model

(3) $r(t) = 608(1.08)^t$ Exponential model

(4) $r(t) = \dfrac{930}{1 + 0.67(1.7)^{-t}}$ Logistic model

a. Which models fit the data significantly better than the rest?

b. Of the models you selected in part (a), which gives the most reasonable prediction for 2010?

---

* Source: Connecticut Division of Special Revenue/*New York Times*, Sept. 23, 2007, p. C1.

using Technology

See the Technology Guides at the end of the chapter for detailed instructions on how to obtain the table and graphs in Example 5 using a TI-83/84 Plus or Excel. Here is an outline:

TI-83/84 Plus

`STAT` EDIT; enter the values of t in L_1, and $r(t)$ in L_2. Turn on Plot1 in Y= screen. `ZOOM` `9`

Adding a curve:

`Y1=-9*X^2+110X+560`

`GRAPH`

[More details on page 112.]

Excel

Table of values: Headings t, Revenue, $r(t)$ in A1–C1; t-values in A2–A8. Revenues in B2–B8. Formula in C2:

`=-9*A2^2+110*A2+560`

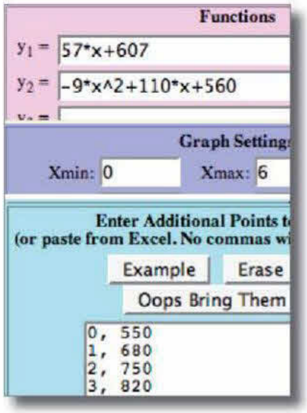
Copy down through C8.

Graph: Highlight A1–C8; insert Scatter chart. [More details on page 118.]

Web Site

www.FiniteandCalc.org

In the Function Evaluator and Grapher utility, enter the data and model(s) as shown below. Set xMin = 0, xMax = 6 and press "Plot Graphs."



Functions

y1 = 57*x+607

y2 = -9*x^2+110*x+560

Graph Setting

Xmin: 0 Xmax: 6

Enter Additional Points t
(or paste from Excel. No commas wi

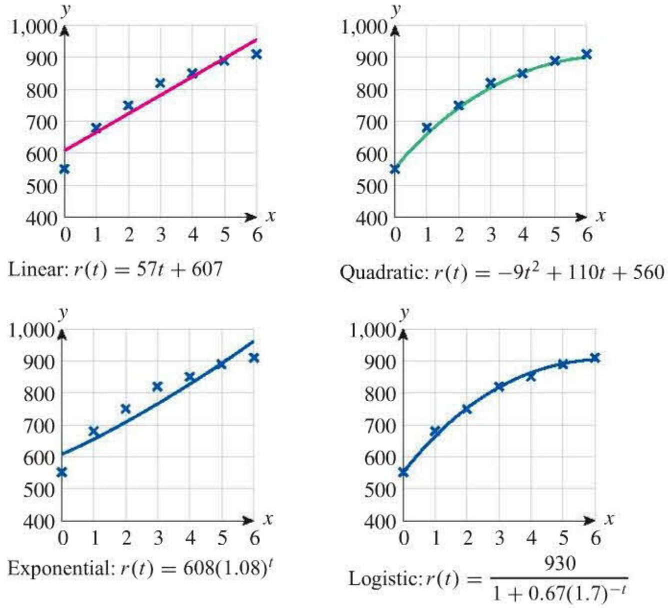
```
0, 550
1, 680
2, 750
3, 820
```

Solution

a. The following table shows the original data together with the values for all four models:

t	0	1	2	3	4	5	6
Revenue ($ million)	550	680	750	820	850	890	910
Linear: $r(t) = 57t + 607$ Technology: `57*x+607`	607	664	721	778	835	892	949
Quadratic: $r(t) = -9t^2 + 110t + 560$ Technology: `-9*x^2+110*x+560`	560	661	744	809	856	885	896
Exponential: $r(t) = 608(1.08)^t$ Technology: `608*(1.08)^x`	608	657	709	766	827	893	965
Logistic: $r(t) = \dfrac{930}{1 + 0.67(1.7)^{-t}}$ Technology: `930/(1+0.67*1.7^(-x))`	557	667	755	818	861	888	905

Notice first that all the values for the quadratic and logistic models are close to the actual revenue values. This cannot be said for the linear and exponential models. Figure 12 shows the original data together with the graph of each model. Notice again that the quadratic and logistic curves appear to follow the data more closely than the other two. We therefore conclude that the quadratic and logistic models fit the data significantly better than the others.



Linear: $r(t) = 57t + 607$

Quadratic: $r(t) = -9t^2 + 110t + 560$

Exponential: $r(t) = 608(1.08)^t$

Logistic: $r(t) = \dfrac{930}{1 + 0.67(1.7)^{-t}}$

Figure 12

b. Although the quadratic and logistic models both appear to fit the data well, they do not both extrapolate to give reasonable predictions for 2010:

$$\text{Quadratic Model: } r(10) = -9(10)^2 + 110(10) + 560 \approx 760$$

$$\text{Logistic Model: } r(10) = \frac{930}{1 + 0.67(1.7)^{-10}} \approx 927$$

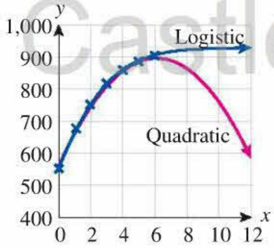


Figure 13

Notice that the quadratic model predicts a significant *decline* in casino revenues whereas the logistic model predicts a more reasonable modest increase. This discrepancy can be seen quite dramatically in Figure 13.

We now derive an analytical model of change over time based on the idea of **compound interest**. Suppose you invest \$500 (the **present value**) in an investment account with an annual yield of 15%, and the interest is reinvested at the end of every year (we say that the interest is **compounded** or **reinvested** once a year). Let t represent the number of years since you made the initial \$500 investment. Each year, the investment is worth 115% (or 1.15 times) of its value the previous year. The **future value** A of your investment changes over time t, so we think of A as a function of t. The following table illustrates how we can calculate the future value for several values of t:

t	0	1	2	3
Future Value $A(t)$	500	575	661.25	760.44
	A	$500(1.15)$	$500(1.15)^2$	$500(1.15)^3$
		×1.15	×1.15	×1.15

Thus, $A(t) = 500(1.15)^t$. A traditional way to write this formula is

$$A(t) = P(1+r)^t$$

where P is the present value ($P = 500$) and r is the annual interest rate ($r = 0.15$).

If, instead of compounding the interest once a year, we compound it every three months (four times a year), we would earn one quarter of the interest ($r/4$ of the current investment) every three months. Because this would happen $4t$ times in t years, the formula for the future value becomes

$$A(t) = P\left(1+\frac{r}{4}\right)^{4t}.$$

Compound Interest

If an amount **(present value)** P is invested for t years at an annual rate of r, and if the interest is compounded (reinvested) m times per year, then the **future value** A is

$$A(t) = P\left(1+\frac{r}{m}\right)^{mt}.$$

A special case is **interest compounded once a year:**

$$A(t) = P(1+r)^t.$$

Quick Example

If \$2,000 is invested for two and a half years in a mutual fund with an annual yield of 12.6% and the earnings are reinvested each month, then $P = 2{,}000$, $r = 0.126$, $m = 12$, and $t = 2.5$, which gives

$$A(2.5) = 2{,}000\left(1+\frac{0.126}{12}\right)^{12\times 2.5}$$

`2000*(1+0.126/12)^(12*2.5)`

$$= 2{,}000(1.0105)^{30} = \$2{,}736.02.$$

EXAMPLE 6 Compound Interest: Investments

Consider the scenario in the preceding Quick Example: You invest \$2,000 in a mutual fund with an annual yield of 12.6% and the interest is reinvested each month.

a. Find the associated exponential model.

b. T Use a table of values to estimate the year during which the value of your investment reaches \$5,000.

c. Use a graph to confirm your answer in part (b).

Solution

a. Apply the formula

$$A(t) = P\left(1 + \frac{r}{m}\right)^{mt}$$

with $P = 2{,}000$, $r = 0.126$, and $m = 12$. We get

$$A(t) = 2{,}000\left(1 + \frac{0.126}{12}\right)^{12t}$$

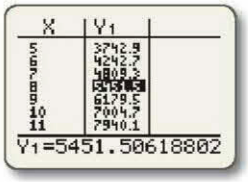
$$A(t) = 2{,}000(1.0105)^{12t}$$ `2000*(1+0.126/12)^(12*t)`

This is the exponential model. (What would happen if we left out the last set of parentheses in the technology formula?)

b. We need to find the value of t for which $A(t) = \$5{,}000$, so we need to solve the equation

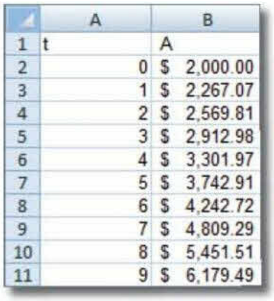
$$5{,}000 = 2{,}000(1.0105)^{12t}$$

In Section 2.3 we will learn how to use logarithms to do this algebraically, but we can answer the question now using a graphing calculator, a spreadsheet, or the Function Evaluator and Grapher utility at the Web site. Just enter the model and compute the balance at the end of several years. Here are examples of tables obtained using three forms of technology:



```
 X    | Y1     |
------+--------+
 5    | 3742.9 |
 6    | 4242.7 |
 7    | 4809.3 |
 8    | 5451.5 |
 9    | 6179.5 |
 10   | 7004.7 |
 11   | 7940.1 |
Y1=5451.50618802
```

TI-83/84 Plus



	A	B
1	t	A
2	0	\$ 2,000.00
3	1	\$ 2,267.07
4	2	\$ 2,569.81
5	3	\$ 2,912.98
6	4	\$ 3,301.97
7	5	\$ 3,742.91
8	6	\$ 4,242.72
9	7	\$ 4,809.29
10	8	\$ 5,451.51
11	9	\$ 6,179.49

Excel

x-Values	y_1-Values
3	2912.98
4	3301.97
5	3742.91
6	4242.72
7	4809.29
8	5451.51

Web site

Because the balance first exceeds \$5,000 at $t = 8$ (the end of year 8) your investment has reached \$5,000 during year 8.

using Technology

TI-83/84 Plus

`Y1=2000(1+0.126/12)^(12X)`

`2ND` `TABLE`

Excel

Headings *t* and *A* in A1–B1; *t*-values 0–11 in A2–A13.

`=2000*(1+0.126/12)^(12*A2)`

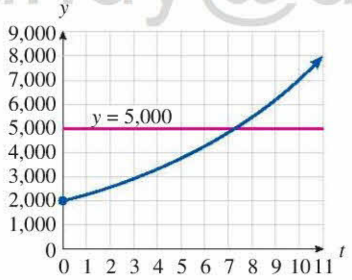
in B2; copy down through B13.

Web Site

www.FiniteandCalc.org

Go to the Function Evaluator and Grapher under Online Utilities, and enter `2000(1+0.126/12)^(12x)` for y_1. Scroll down to the Evaluator, enter the values 0–11 under *x*-values and press "Evaluate."

c. Figure 14 shows the graph of $A(t) = 2{,}000(1.0105)^{12t}$ together with the horizontal line $y = 5{,}000$.



Technology format: `2000*1.0105^(12*x)`

Figure 14

The graphs cross between $t = 7$ and $t = 8$, confirming that year 8 is the first year during which the value of the investment reaches $5,000.

The compound interest examples we saw above are instances of **exponential growth:** a quantity whose magnitude is an increasing exponential function of time. The decay of unstable radioactive isotopes provides instances of **exponential decay:** a quantity whose magnitude is a *decreasing* exponential function of time. For example, carbon 14, an unstable isotope of carbon, decays exponentially to nitrogen. Because carbon 14 decay is extremely slow, it has important applications in the dating of fossils.

EXAMPLE 7 Exponential Decay: Carbon Dating

The amount of carbon 14 remaining in a sample that originally contained A grams is approximately

$$C(t) = A(0.999879)^t$$

where t is time in years.

a. What percentage of the original amount remains after one year? After two years?

b. Graph the function C for a sample originally containing 50g of carbon 14, and use your graph to estimate how long, to the nearest 1,000 years, it takes for half the original carbon 14 to decay.

c. A fossilized plant unearthed in an archaeological dig contains 0.50 gram of carbon 14 and is known to be 50,000 years old. How much carbon 14 did the plant originally contain?

Solution Notice that the given model is exponential as it has the form $f(t) = Ab^t$. (See page 51.)

a. At the start of the first year, $t = 0$, so there are

$$C(0) = A(0.999879)^0 = A \text{ grams.}$$

At the end of the first year, $t = 1$, so there are

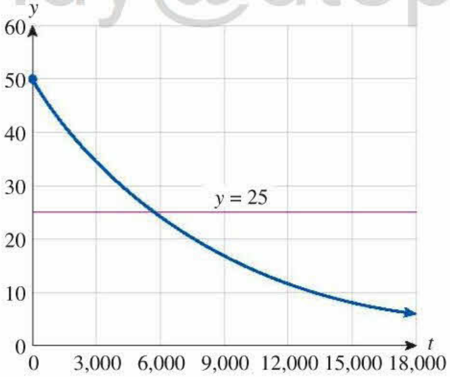
$$C(1) = A(0.999879)^1 = 0.999879A \text{ grams;}$$

that is, 99.9879% of the original amount remains. After the second year, the amount remaining is

$$C(2) = A(0.999879)^2 \approx 0.999758A \text{ grams,}$$

or about 99.9758% of the original sample.

b. For a sample originally containing 50g of carbon 14, $A = 50$, so $C(t) = 50(0.999879)^t$. Its graph is shown in Figure 15.



Technology format: `50*0.999879^x`

Figure 15

We have also plotted the line $y = 25$ on the same graph. The graphs intersect at the point where the original sample has decayed to 25g: about $t = 6{,}000$ years.

c. We are given the following information: $C = 0.50$, $A =$ the unknown, and $t = 50{,}000$. Substituting gives

$$0.50 = A(0.999879)^{50{,}000}.$$

Solving for A gives

$$A = \frac{0.5}{0.999879^{50{,}000}} \approx 212 \text{ grams.}$$

Thus, the plant originally contained 212 grams of carbon 14.

➡ **Before we go on...** The formula we used for A in Example 7(c) has the form

$$A(t) = \frac{C}{0.999879^t},$$

which gives the original amount of carbon 14 t years ago in terms of the amount C that is left now. A similar formula can be used in finance to find the present value, given the future value. ■

1.2 EXERCISES

▼ more advanced ◆ challenging

T indicates exercises that should be solved using technology

1. ***Resources*** You now have 200 sound files on your hard drive, and this number is increasing by 10 sound files each day. Find a mathematical model for this situation. HINT [See Quick Example 1 on page 56.]

2. ***Resources*** The amount of free space left on your hard drive is now 50 gigabytes (GB) and is decreasing by 5 GB/month. Find a mathematical model for this situation.

3. ***Soccer*** My rectangular soccer field site has a length equal to twice its width. Find its area in terms of its length x. HINT [See Quick Example 3 on page 56.]

4. ***Cabbage*** My rectangular cabbage patch has a total area of 100 sq. ft. Find its perimeter in terms of the width x.

5. ***Vegetables*** I want to fence in a square vegetable patch. The fencing for the east and west sides costs \$4 per foot, and the fencing for the north and south sides costs only \$2 per foot. Find the total cost of the fencing as a function of the length of a side x.

6. ***Orchids*** My square orchid garden abuts my house so that the house itself forms the northern boundary. The fencing for the southern boundary costs \$4 per foot, and the fencing for the east and west sides costs \$2 per foot. Find the total cost of the fencing as a function of the length of a side x.

7. ***Cost*** A piano manufacturer has a daily fixed cost of \$1,200 and a marginal cost of \$1,500 per piano. Find the cost $C(x)$ of manufacturing x pianos in one day. Use your function to answer the following questions:

a. On a given day, what is the cost of manufacturing 3 pianos?
b. What is the cost of manufacturing the 3rd piano that day?

c. What is the cost of manufacturing the 11th piano that day?

d. What is the variable cost? What is the fixed cost? What is the marginal cost?

e. Graph C as a function of x. HINT [See Example 1.]

8. ***Cost*** The cost of renting tuxes for the Choral Society's formal is \$20 down, plus \$88 per tux. Express the cost C as a function of x, the number of tuxedos rented. Use your function to answer the following questions.

a. What is the cost of renting 2 tuxes?

b. What is the cost of the 2nd tux?

c. What is the cost of the 4,098th tux?

d. What is the variable cost? What is the marginal cost?

e. Graph C as a function of x.

9. ***Break-Even Analysis*** Your college newspaper, *The Collegiate Investigator*, has fixed production costs of \$70 per edition and marginal printing and distribution costs of 40¢ per copy. *The Collegiate Investigator* sells for 50¢ per copy.

a. Write down the associated cost, revenue, and profit functions. HINT [See Examples 1 and 2.]

b. What profit (or loss) results from the sale of 500 copies of *The Collegiate Investigator*?

c. How many copies should be sold in order to break even?

10. ***Break-Even Analysis*** The Audubon Society at Enormous State University (ESU) is planning its annual fund-raising "Eat-a-thon." The society will charge students 50¢ per serving of pasta. The only expenses the society will incur are the cost of the pasta, estimated at 15¢ per serving, and the \$350 cost of renting the facility for the evening.

a. Write down the associated cost, revenue, and profit functions.

b. How many servings of pasta must the Audubon Society sell in order to break even?

c. What profit (or loss) results from the sale of 1,500 servings of pasta?

11. ***Break-Even Analysis*** Gymnast Clothing manufactures expensive hockey jerseys for sale to college bookstores in runs of up to 200. Its cost (in dollars) for a run of x hockey jerseys is

$$C(x) = 2{,}000 + 10x + 0.2x^2 \quad (0 \le x \le 200)$$

Gymnast Clothing sells the jerseys at \$100 each. Find the revenue and profit functions. How many should Gymnast Clothing manufacture to make a profit? HINT [See Example 2.]

12. ***Break-Even Analysis*** Gymnast Clothing also manufactures expensive soccer cleats for sale to college bookstores in runs of up to 500. Its cost (in dollars) for a run of x pairs of cleats is

$$C(x) = 3{,}000 + 8x + 0.1x^2 \quad (0 \le x \le 500)$$

Gymnast Clothing sells the cleats at \$120 per pair. Find the revenue and profit functions. How many should Gymnast Clothing manufacture to make a profit?

13. ***Break-Even Analysis: School Construction Costs*** The cost, in millions of dollars, of building a two-story high school in New York State was estimated to be

$$C(x) = 1.7 + 0.12x - 0.0001x^2 \quad (20 \le x \le 400)$$

where x is the number of thousands of square feet.[12] Suppose that you are contemplating building a for-profit two-story high school and estimate that your total revenue will be \$0.1 million dollars per thousand square feet. What is the profit function? What size school should you build in order to break even?

14. ***Break-Even Analysis: School Construction Costs*** The cost, in millions of dollars, of building a three-story high school in New York State was estimated to be

$$C(x) = 1.7 + 0.14x - 0.0001x^2 \quad (20 \le x \le 400)$$

where x is the number of thousands of square feet.[13] Suppose that you are contemplating building a for-profit three-story high school and estimate that your total revenue will be \$0.2 million dollars per thousand square feet. What is the profit function? What size school should you build in order to break even?

15. ▼ ***Profit Analysis—Aviation*** The hourly operating cost of a **Boeing** 747-100, which seats up to 405 passengers, is estimated to be \$5,132.[14] If an airline charges each passenger a fare of \$100 per hour of flight, find the hourly profit P it earns operating a 747-100 as a function of the number of passengers x. (Be sure to specify the domain.) What is the least number of passengers it must carry in order to make a profit? HINT [The cost function is constant (Variable cost = 0).]

16. ▼ ***Profit Analysis—Aviation*** The hourly operating cost of a **McDonnell Douglas** DC 10-10, which seats up to 295 passengers, is estimated to be \$3,885.[15] If an airline charges each passenger a fare of \$100 per hour of flight, find the hourly profit P it earns operating a DC 10-10 as a function of the number of passengers x. (Be sure to specify the domain.) What is the least number of passengers it must carry in order to make a profit? HINT [The cost function is constant (Variable cost = 0).]

17. ▼ ***Break-Even Analysis*** *(based on a question from a CPA exam)* The Oliver Company plans to market a new product. Based on its market studies, Oliver estimates that it can sell up to 5,500 units in 2005. The selling price will be \$2 per unit. Variable costs are estimated to be 40% of total revenue. Fixed costs are estimated to be \$6,000 for 2005. How many units should the company sell to break even?

18. ▼ ***Break-Even Analysis*** *(based on a question from a CPA exam)* The Metropolitan Company sells its latest product at a unit price of \$5. Variable costs are estimated to be 30% of the total revenue, while fixed costs amount to \$7,000 per month. How many units should the company sell per month in order to break even, assuming that it can sell up to 5,000 units per month at the planned price?

---

[12] The model is the authors'. Source for data: *Project Labor Agreements and Public Construction Cost in New York State*, Paul Bachman and David Tuerck, Beacon Hill Institute at Suffolk University, April 2006, www.beaconhill.org.

[13] Ibid.

[14] In 1992. Source: Air Transportation Association of America.

[15] Ibid.

19. ◆ ***Break-Even Analysis*** *(from a CPA exam)* Given the following notations, write a formula for the break-even sales level.

SP = Selling price per unit
FC = Total fixed cost
VC = Variable cost per unit

20. ◆ ***Break-Even Analysis*** *(based on a question from a CPA exam)* Given the following notation, give a formula for the total fixed cost.

SP = Selling price per unit
VC = Variable cost per unit
BE = Break even sales level in units

21. ◆ ***Break-Even Analysis—Organized Crime*** The organized crime boss and perfume king Butch (Stinky) Rose has daily overheads (bribes to corrupt officials, motel photographers, wages for hit men, explosives, and so on) amounting to \$20,000 per day. On the other hand, he has a substantial income from his counterfeit perfume racket: He buys imitation French perfume (Chanel № 22.5) at \$20 per gram, pays an additional \$30 per 100 grams for transportation, and sells it via his street thugs for \$600 per gram. Specify Stinky's profit function, $P(x)$, where x is the quantity (in grams) of perfume he buys and sells, and use your answer to calculate how much perfume should pass through his hands per day in order that he break even.

22. ◆ ***Break-Even Analysis—Disorganized Crime*** Butch (Stinky) Rose's counterfeit Chanel № 22.5 racket has run into difficulties; it seems that the *authentic* Chanel № 22.5 perfume is selling for less than his counterfeit perfume. However, he has managed to reduce his fixed costs to zero, and his overall costs are now \$400 per gram plus \$30 per gram transportation costs and commission. (The perfume's smell is easily detected by specially trained Chanel Hounds, and this necessitates elaborate packaging measures.) He therefore decides to sell it for \$420 per gram in order to undercut the competition. Specify Stinky's profit function, $P(x)$, where x is the quantity (in grams) of perfume he buys and sells, and use your answer to calculate how much perfume should pass through his hands per day in order that he break even. Interpret your answer.

23. ***Demand for Monorail Service, Las Vegas*** The demand for monorail service in Las Vegas can be approximated by

$$q(p) = 64p^{-0.76} \text{ thousand rides per day} \quad (3 \le p \le 5)$$

where p is the cost per ride in dollars.[16]

a. Graph the demand function.
b. What is the result on demand if the cost per ride is increased from \$3.00 to \$3.50? HINT [See Example 3.]

24. ***Demand for Monorail Service, Mars*** The demand for monorail service on the Utarek monorail, which links the three urbynes (or districts) of Utarek, Mars, can be approximated by

$$q(p) = 30p^{-0.49} \quad \text{million rides per day} \quad (3 \le p \le 5)$$

where p is the cost per ride in zonars ($\bar{\bar{\mathbf{Z}}}$).[17]

a. Graph the demand function.
b. What is the result on demand if the cost per ride is decreased from $\bar{\bar{\mathbf{Z}}}$5.00 to $\bar{\bar{\mathbf{Z}}}$3.50?

25. ▼ ***Demand*** The demand for Sigma Mu Fraternity plastic brownie dishes is

$$q(p) = 361{,}201 - (p+1)^2$$

where q represents the number of brownie dishes Sigma Mu can sell each month at a price of p¢. Use this function to determine:

a. The number of brownie dishes Sigma Mu can sell each month if the price is set at 50¢.
b. The number of brownie dishes they can unload each month if they give them away.
c. The lowest price at which Sigma Mu will be unable to sell any dishes.

26. ▼ ***Revenue*** The total weekly revenue earned at Royal Ruby Retailers is given by

$$R(p) = -\frac{4}{3}p^2 + 80p$$

where p is the price (in dollars) RRR charges per ruby. Use this function to determine:

a. The weekly revenue, to the nearest dollar, when the price is set at \$20/ruby.
b. The weekly revenue, to the nearest dollar, when the price is set at \$200/ruby. (Interpret your result.)
c. The price RRR should charge in order to obtain a weekly revenue of \$1,200.

27. ***Equilibrium Price*** The demand for your hand-made skateboards, in weekly sales, is $q = -3p + 700$ if the selling price is \$$p$. You are prepared to supply $q = 2p - 500$ per week at the price \$$p$. At what price should you sell your skateboards so that there is neither a shortage nor a surplus? HINT [See Quick Example on page 62.]

28. ***Equilibrium Price*** The demand for your factory-made skateboards, in weekly sales, is $q = -5p + 50$ if the selling price is \$$p$. If you are selling them at that price, you can obtain $q = 3p - 30$ per week from the factory. At what price should you sell your skateboards so that there is neither a shortage nor a surplus?

29. ***Equilibrium Price: Cell Phones*** Worldwide quarterly sales of Nokia® cell phones was approximately $q = -p + 156$ million phones when the wholesale price[18] was \$$p$.

a. If Nokia was prepared to supply $q = 4p - 394$ million phones per quarter at a wholesale price of \$$p$, what would be the equilibrium price?
b. The actual wholesale price was \$105 in the fourth quarter of 2004. Estimate the projected shortage or surplus at that price. HINT [See Quick Example on page 62 and also Example 4.]

---

[16] Source: *New York Times*, Februrary 10, 2007, p. A9.

[17] $\bar{\bar{\mathbf{Z}}}$ designates Zonars, the official currency of Mars. See www.marsnext.com for details of the Mars colony, its commerce, and its culture.

[18] Source: Embedded.com/Company reports December, 2004.

30. ***Equilibrium Price: Cell Phones*** Worldwide annual sales of all cell phones is approximately $-10p + 1{,}600$ million phones when the wholesale price[19] is \$$p$.

a. If manufacturers are prepared to supply $q = 14p - 800$ million phones per year at a wholesale price of \$$p$, what would be the equilibrium price?

b. The actual wholesale price was projected to be \$80 in the fourth quarter of 2008. Estimate the projected shortage or surplus at that price.

31. T ***Demand for Monorail Service, Las Vegas*** The demand for monorail service in Las Vegas can be modeled by

$$q = 64p^{-0.76} \text{ thousand rides per day}$$

where p is the fare the Las Vegas Monorail Company charges in dollars.[20] Assume the company is prepared to provide service for

$$q = 2.5p + 15.5 \text{ thousand rides per day}$$

at a fare of \$$p$.

a. Graph the demand and supply equations, and use your graph to estimate the equilibrium price (to the nearest 50¢).

b. Estimate, to the nearest 10 rides, the shortage or surplus of monorail service at the December 2005 fare of \$5 per ride.

32. T ***Demand for Monorail Service, Mars*** The demand for monorail service in the three urbynes (or districts) of Utarek, Mars can be modeled by

$$q = 31p^{-0.49} \text{ million rides per day}$$

where p is the fare the Utarek Monorail Cooperative charges in zonars (Ƶ).[21] Assume the cooperative is prepared to provide service for

$$q = 2.5p + 17.5 \text{ thousand rides per day}$$

at a fare of Ƶp.

a. Graph the demand and supply equations, and use your graph to estimate the equilibrium price (to the nearest 0.50 zonars).

b. Estimate the shortage or surplus of monorail service at the December 2085 fare of Ƶ1 per ride.

33. ▼ ***Toxic Waste Treatment*** The cost of treating waste by removing PCPs goes up rapidly as the quantity of PCPs removed goes up. Here is a possible model:

$$C(q) = 2{,}000 + 100q^2$$

where q is the reduction in toxicity (in pounds of PCPs removed per day) and $C(q)$ is the daily cost (in dollars) of this reduction.

a. Find the cost of removing 10 pounds of PCPs per day.

b. Government subsidies for toxic waste cleanup amount to

$$S(q) = 500q$$

where q is as above and $S(q)$ is the daily dollar subsidy. Calculate the net cost function $N(q)$ (the cost of removing q pounds of PCPs per day after the subsidy is taken into account), given the cost function and subsidy above, and find the net cost of removing 20 pounds of PCPs per day.

34. ▼ ***Dental Plans*** A company pays for its employees' dental coverage at an annual cost C given by

$$C(q) = 1{,}000 + 100\sqrt{q}$$

where q is the number of employees covered and $C(q)$ is the annual cost in dollars.

a. If the company has 100 employees, find its annual outlay for dental coverage.

b. Assuming that the government subsidizes coverage by an annual dollar amount of

$$S(q) = 200q$$

calculate the net cost function $N(q)$ to the company, and calculate the net cost of subsidizing its 100 employees. Comment on your answer.

35. ***Spending on Corrections in the 1990s*** The following table shows the annual spending by all states in the United States on corrections ($t = 0$ represents the year 1990):[22]

Year (t)	0	2	4	6	7
Spending (\$ billion)	16	18	22	28	30

a. Which of the following functions best fits the given data? (*Warning*: None of them fits exactly, but one fits more closely than the others.) HINT [See Example 5.]

(A) $S(t) = -0.2t^2 + t + 16$
(B) $S(t) = 0.2t^2 + t + 16$
(C) $S(t) = t + 16$

b. Use your answer to part (a) to "predict" spending on corrections in 1998, assuming that the trend continued.

36. ***Spending on Corrections in the 1990s*** Repeat Exercise 35, this time choosing from the following functions:

a. $S(t) = 16 + 2t$
b. $S(t) = 16 + t + 0.5t^2$
c. $S(t) = 16 + t - 0.5t^2$

Freon Production *Exercises 37 and 38 are based on the following data in Quick Example 1 on page 42 showing the amount of ozone-damaging Freon (in tons) produced in developing countries in year t since 2000:*

---

[19] Wholesale price projections are the authors'. Source for sales prediction: I-Stat/NDR December, 2004.

[20] The model is the authors'. Source for data: *New York Times*, Februrary 10, 2007, p. A9.

[21] Ƶ designates Zonars, the official currency of Mars. See www.marsnext.com for details of the Mars colony, its commerce, and its culture.

[22] Data are rounded. Source: National Association of State Budget Officers/*The New York Times*, February 28, 1999, p. A1.

t (Year Since 2000)	*F* (Tons of Freon 22)
0	100
2	140
4	200
6	270
8	400
10	590

Source: *New York Times*, February 23, 2007, p. C1

37. a. Which two of the following models best fit the given data?

(A) $f(t) = 98(1.2^t)$
(B) $f(t) = 4.6t^2 + 1.2t + 109$
(C) $f(t) = 47t + 48$
(D) $f(t) = 98(1.2^{-t})$

b. Of the two models you chose in part (a), which predicts the larger amount of freon in 2020? How much freon does that model predict?

38. Repeat Exercise 37 using the following models:

a. $f(t) = -4.6t^2 + 1.2t + 109$
b. $f(t) = \dfrac{2500}{1 + 22(1.2^{-t})}$
c. $f(t) = 65t - 60$
d. $f(t) = 4.6t^2 + 1.2t + 109$

39. *Soccer Gear* The East Coast College soccer team is planning to buy new gear for its road trip to California. The cost per shirt depends on the number of shirts the team orders as shown in the following table:

x (Shirts ordered)	5	25	40	100	125
A(*x*) (Cost/shirt, $)	22.91	21.81	21.25	21.25	22.31

a. Which of the following functions best models the data?

(A) $A(x) = 0.005x + 20.75$
(B) $A(x) = 0.01x + 20 + \frac{25}{x}$
(C) $A(x) = 0.0005x^2 - 0.07x + 23.25$
(D) $A(x) = 25.5(1.08)^{(x-5)}$

b. **T** Graph the model you chose in part (a) for $10 \le x \le 100$. Use your graph to estimate the lowest cost per shirt and the number of shirts the team should order to obtain the lowest price per shirt.

40. *Hockey Gear* The South Coast College hockey team wants to purchase wool hats for its road trip to Alaska. The cost per hat depends on the number of hats the team orders as shown in the following table:

x (Hats ordered)	5	25	40	100	125
A(*x*) (Cost/hat, $)	25.50	23.50	24.63	30.25	32.70

a. Which of the following functions best models the data?

(A) $A(x) = 0.05x + 20.75$
(B) $A(x) = 0.1x + 20 + \frac{25}{x}$
(C) $A(x) = 0.0008x^2 - 0.07x + 23.25$
(D) $A(x) = 25.5(1.08)^{(x-5)}$

b. **T** Graph the model you chose in part (a) with $5 \le x \le 30$. Use your graph to estimate the lowest cost per hat and the number of hats the team should order to obtain the lowest price per hat.

41. *Value of Euro* The following table shows the approximate value V of one Euro in U.S. dollars from its introduction in January 2000 to January 2008. ($t = 0$ represents January 2000.)[23]

t (Year)	0	2	8
V (Value in $)	1.00	0.90	1.40

Which of the following kinds of models would best fit the given data? Explain your choice of model. (A, a, b, c, and m are constants.)

(A) Linear: $V(t) = mt + b$
(B) Quadratic: $V(t) = at^2 + bt + c$
(C) Exponential: $V(t) = Ab^t$

42. *Household Income* The following table shows the approximate average household income in the United States in 1990, 1995, and 2003. ($t = 0$ represents 1990.)[24]

t (Year)	0	5	13
H (Household Income in $1,000)	30	35	43

Which of the following kinds of models would best fit the given data? Explain your choice of model. (A, a, b, c, and m are constants.)

(A) Linear: $H(t) = mt + b$
(B) Quadratic: $H(t) = at^2 + bt + c$
(C) Exponential: $H(t) = Ab^t$

43. *Investments* In 2007, **E*TRADE Financial** was offering 4.94% interest on its online savings accounts, with interest reinvested monthly.[25] Find the associated exponential model for the value of a $5,000 deposit after t years. Assuming this rate of return continued for 7 years, how much would a deposit of $5,000 at the beginning of 2007 be worth at the start of 2014? (Answer to the nearest $1.)

---

[23] Source: Bloomberg Financial Markets.

[24] In current dollars, unadjusted for inflation. Source: U.S. Census Bureau; "Table H-5. Race and Hispanic Origin of Householder—Households by Median and Mean Income: 1967 to 2003"; published August 27, 2004; www.census.gov/hhes/income/histinc/h05.html.

[25] Interest rate based on annual percentage yield. Source: www.us.etrade.com, December 2007.

44. ***Investments*** In 2007, **ING Direct** was offering 4.14% interest on its online Orange Savings Account, with interest reinvested quarterly.[26] Find the associated exponential model for the value of a \$4,000 deposit after t years. Assuming this rate of return continued for eight years, how much would a deposit of \$4,000 at the beginning of 2007 be worth at the start of 2015? (Answer to the nearest \$1.)

45. T ***Investments*** Refer to Exercise 43. At the start of which year will an investment of \$5,000 made at the beginning of 2007 first exceed \$7,500?

46. T ***Investments*** Refer to Exercise 44. In which year will an investment of \$4,000 made at the beginning of 2007 first exceed \$6,000?

47. ***Carbon Dating*** A fossil originally contained 104 grams of carbon 14. Refer to the formula for $C(t)$ in Example 7 and estimate the amount of carbon 14 left in the sample after 10,000 years, 20,000 years, and 30,000 years. HINT [See Example 7.]

48. ***Carbon Dating*** A fossil contains 4.06 grams of carbon 14. Refer to the formula for $C(t)$ in Example 7 and estimate the amount of carbon 14 in the sample 10,000 years, 20,000 years, and 30,000 years ago.

49. ***Carbon Dating*** A fossil contains 4.06 grams of carbon 14. It is estimated that the fossil originally contained 46 grams of carbon 14. By calculating the amount left after 5,000 years, 10,000 years, . . . , 35,000 years, estimate the age of the sample to the nearest 5,000 years. (Refer to the formula for $C(t)$ in Example 7.)

50. ***Carbon Dating*** A fossil contains 2.8 grams of carbon 14. It is estimated that the fossil originally contained 104 grams of carbon 14. By calculating the amount 5,000 years, 10,000 years, . . . , 35,000 years ago, estimate the age of the sample to the nearest 5,000 years. (Refer to the formula for $C(t)$ in Example 7.)

51. ***Radium Decay*** The amount of radium 226 remaining in a sample that originally contained A grams is approximately

$$C(t) = A(0.999567)^t$$

where t is time in years.

a. Find, to the nearest whole number, the percentage of iodine-131 left in an originally pure sample after 1,000 years, 2,000 years, and 3,000 years.

b. Use a graph to estimate, to the nearest 100 years, when one half of a sample of 100 grams will have decayed.

52. ***Iodine Decay*** The amount of iodine 131 remaining in a sample that originally contained A grams is approximately

$$C(t) = A(0.9175)^t$$

where t is time in days.

a. Find, to the nearest whole number, the percentage of iodine 131 left in an originally pure sample after 2 days, 4 days, and 6 days.

b. Use a graph to estimate, to the nearest day, when one half of a sample of 100 grams will have decayed.

COMMUNICATION AND REASONING EXERCISES

53. If the population of the lunar station at Clavius has a population of $P = 200 + 30t$ where t is time in years since the station was established, then the population is increasing by ____ per year.

54. My bank balance can be modeled by $B(t) = 5{,}000 - 200t$ dollars, where t is time in days since I opened the account. The balance on my account is __________ by \$200 per day.

55. Classify the following model as analytical or curve-fitting, and give a reason for your choice: The price of gold was \$700 on Monday, \$710 on Tuesday, and \$700 on Wednesday. Therefore, the price can be modeled by $p(t) = -10t^2 + 20t + 700$ where t is the day since Monday.

56. Classify the following model as analytical or curve-fitting, and give a reason for your choice: The width of a small animated square on my computer screen is currently 10 mm and is growing by 2 mm per second. Therefore, its area can be modeled by $a(t) = (10 + 2t)^2$ square mm where t is time in seconds.

57. Fill in the missing information for the following *analytical model* (answers may vary): _____________. Therefore, the cost of downloading a movie can be modeled by $c(t) = 4 - 0.2t$, where t is time in months since January.

58. Repeat Exercise 57, but this time regard the given model as a *curve-fitting model*.

59. Fill in the blanks: In a linear cost function, the _______ cost is x times the _______ cost.

60. Complete the following sentence: In a linear cost function, the marginal cost is the ______________.

61. ▼ We said on page 61 that the demand for a commodity generally goes down as the price goes up. Assume that the demand for a certain commodity goes up as the price goes up. Is it still possible for there to be an equilibrium price? Explain with the aid of a demand and supply graph.

62. ▼ What would happen to the price of a certain commodity if the demand was always greater than the supply? Illustrate with a demand and supply graph.

63. You have a set of data points showing the sales of videos on your Web site versus time that are closely approximated by two different mathematical models. Give one criterion that would lead you to choose one over the other. (Answers may vary.)

64. Would it ever be reasonable to use a quadratic model $s(t) = at^2 + bt + c$ to predict long-term sales if a is negative? Explain.

---

[26] Interest rate based on annual percentage yield. Source: www.home.ingdirect.com, December 2007.

1.3 Linear Functions and Models

Linear functions are among the simplest functions and are perhaps the most useful of all mathematical functions.

Linear Function

A **linear function** is one that can be written in the form

$f(x) = mx + b$ Function form

or

$y = mx + b$ Equation form

Quick Example

$f(x) = 3x - 1$

$y = 3x - 1$

where m and b are fixed numbers. (The names m and b are traditional.*)

*** NOTE** Actually, c is sometimes used instead of b. As for m, there has even been some research lately into the question of its origin, but no one knows exactly why the letter m is used.

Linear Functions from the Numerical and Graphical Point of View

The following table shows values of $y = 3x - 1$ ($m = 3$, $b = -1$) for some values of x:

x	-4	-3	-2	-1	0	1	2	3	4
y	-13	-10	-7	-4	-1	2	5	8	11

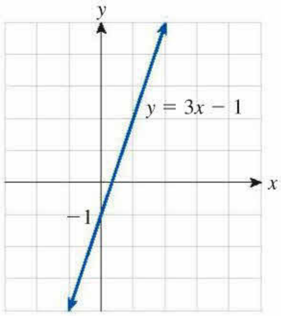


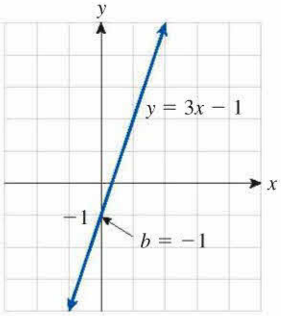
Figure 16

Its graph is shown in Figure 16.

Looking first at the table, notice that that setting $x = 0$ gives $y = -1$, the value of b.

Numerically, b is the value of y when x = 0.

On the graph, the corresponding point $(0, -1)$ is the point where the graph crosses the y-axis, and we say that $b = -1$ is the ***y*-intercept** of the graph (Figure 17).



y-intercept $= b = -1$
Graphically, b is the y-intercept of the graph.

Figure 17

What about m? Looking once again at the table, notice that y increases by $m = 3$ units for every increase of 1 unit in x. This is caused by the term $3x$ in the formula: For every increase of 1 in x, we get an increase of $3 \times 1 = 3$ in y.

Numerically, y increases by m units for every 1-unit increase of x.

Likewise, for every increase of 2 in x we get an increase of $3 \times 2 = 6$ in y. In general, if x increases by some amount, y will increase by three times that amount. We write:

Change in $y = 3 \times$ Change in x.

The Change in a Quantity: Delta Notation

If a quantity q changes from q_1 to q_2, the **change in q** is just the difference:

$$\begin{aligned}\text{Change in } q &= \text{Second value} - \text{First value}\\ &= q_2 - q_1\end{aligned}$$

Mathematicians traditionally use Δ (delta, the Greek equivalent of the Roman letter D) to stand for change, and write the change in q as Δq.

$$\Delta q = \text{Change in } q = q_2 - q_1$$

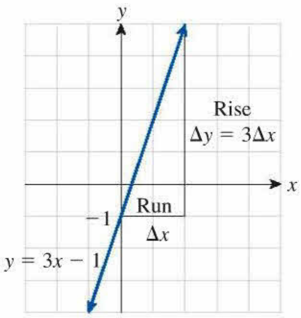
Quick Examples

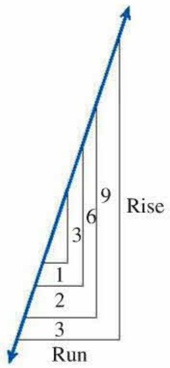
1. If x is changed from 1 to 3, we write

$$\Delta x = \text{Second value} - \text{First value} = 3 - 1 = 2.$$

2. Looking at our linear function, we see that when x changes from 1 to 3, y changes from 2 to 8. So,

$$\Delta y = \text{Second value} - \text{First value} = 8 - 2 = 6.$$





Slope = $m = 3$
Graphically, m is the slope of the graph.

Figure 18

Using delta notation, we can now write, for our linear function,

$$\Delta y = 3\Delta x \qquad \text{Change in } y = 3 \times \text{Change in } x.$$

or

$$\frac{\Delta y}{\Delta x} = 3.$$

Because the value of y increases by exactly 3 units for every increase of 1 unit in x, the graph is a straight line rising by 3 units for every 1 unit we go to the right. We say that we have a **rise** of 3 units for each **run** of 1 unit. Because the value of y changes by $\Delta y = 3\Delta x$ units for every change of Δx units in x, in general we have a rise of $\Delta y = 3\Delta x$ units for each run of Δx units (Figure 18). Thus, we have a rise of 6 for a run of 2, a rise of 9 for a run of 3, and so on. So, $m = 3$ is a measure of the steepness of the line; we call m the **slope of the line:**

$$\text{Slope} = m = \frac{\Delta y}{\Delta x} = \frac{\text{Rise}}{\text{Run}}$$

In general (replace the number 3 by a general number m), we can say the following.

The Roles of m and b in the Linear Function $f(x) = mx + b$

Role of m

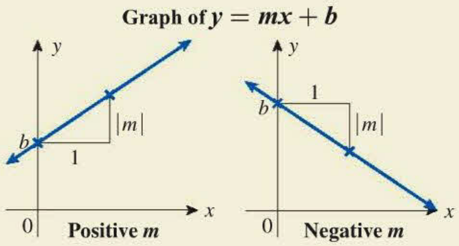
Numerically If $y = mx + b$, then y changes by m units for every 1-unit change in x. A change of Δx units in x results in a change of $\Delta y = m\Delta x$ units in y. Thus,

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{Change in } y}{\text{Change in } x}.$$

Graphically m is the slope of the line $y = mx + b$:

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{Rise}}{\text{Run}} = \text{Slope}.$$

For positive m, the graph rises m units for every 1-unit move to the right, and rises $\Delta y = m\Delta x$ units for every Δx units moved to the right. For negative m, the graph drops $|m|$ units for every 1-unit move to the right, and drops $|m|\Delta x$ units for every Δx units moved to the right.



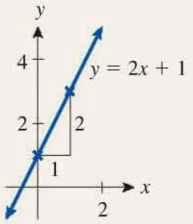
Role of *b*

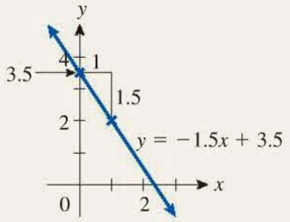
Numerically When $x = 0$, $y = b$.

Graphically b is the y-intercept of the line $y = mx + b$.

Quick Examples

1. $f(x) = 2x + 1$ has slope $m = 2$ and y-intercept $b = 1$. To sketch the graph, we start at the y-intercept $b = 1$ on the y-axis, and then move 1 unit to the right and up $m = 2$ units to arrive at a second point on the graph. Now connect the two points to obtain the graph on the left.





2. The line $y = -1.5x + 3.5$ has slope $m = -1.5$ and y-intercept $b = 3.5$. Because the slope is negative, the graph (above right) goes *down* 1.5 units for every 1 unit it moves to the right.

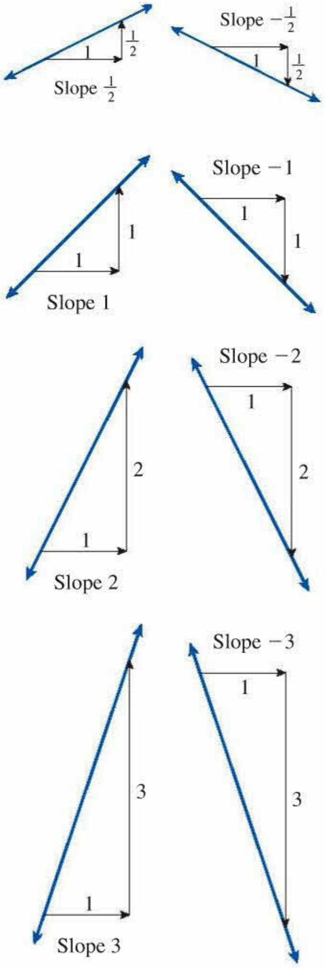


Figure 19

It helps to be able to picture what different slopes look like, as in Figure 19. Notice that the larger the absolute value of the slope, the steeper is the line.

using Technology

See the Technology Guides at the end of the chapter for detailed instructions on how to obtain a table with the successive quotients $m = \Delta y/\Delta x$ for the functions f and g in Example 1 using a TI-83/84 Plus or Excel. These tables show at a glance that f is not linear. Here is an outline:

TI-83/84 Plus

`STAT` EDIT; Enter values of x and $f(x)$ in lists L_1 and L_2. Highlight the heading L_3 and enter the following formula (including the quotes):

`"ΔList(L2)/ΔList(L1)"`

[More details on page 110.]

Excel

Enter headings x, $f(x)$, Df/Dx in cells A1–C1, and the corresponding values from one of the tables in cells A2–B8. Enter `=(B3-B2)/(A3-A2)` in cell C2, and copy down through C8. [More details on page 116.]

EXAMPLE 1 Recognizing Linear Data Numerically and Graphically

Which of the following two tables gives the values of a linear function? What is the formula for that function?

x	0	2	4	6	8	10	12
$f(x)$	3	−1	−3	−6	−8	−13	−15

x	0	2	4	6	8	10	12
$g(x)$	3	−1	−5	−9	−13	−17	−21

Solution The function f cannot be linear: If it were, we would have $\Delta f = m\Delta x$ for some fixed number m. However, although the change in x between successive entries in the table is $\Delta x = 2$ each time, the change in f is not the same each time. Thus, the ratio $\Delta f/\Delta x$ is not the same for every successive pair of points.

On the other hand, the ratio $\Delta g/\Delta x$ is the same each time, namely,

$$\frac{\Delta g}{\Delta x} = \frac{-4}{2} = -2$$

Δx $\quad 2 - 0 = 2 \quad 4 - 2 = 2 \quad 6 - 4 = 2 \quad 8 - 6 = 2 \quad 10 - 8 = 2 \quad 12 - 10 = 2$

x	0	2	4	6	8	10	12
$g(x)$	3	−1	−5	−9	−13	−17	−21

Δg $\quad (-1) - 3 = -4 \quad -5 - (-1) = -4 \quad -9 - (-5) = -4 \quad -13 - (-9) = -4 \quad -17 - (-13) = -4 \quad -21 - (-17) = -4$

Thus, g is linear with slope $m = -2$. By the table, $g(0) = 3$, hence $b = 3$. Thus,

$$g(x) = -2x + 3. \qquad \text{Check that this formula gives the values in the table.}$$

If you graph the points in the tables defining f and g above, it becomes easy to see that g is linear and f is not; the points of g lie on a straight line (with slope −2), whereas the points of f do not lie on a straight line (Figure 20).

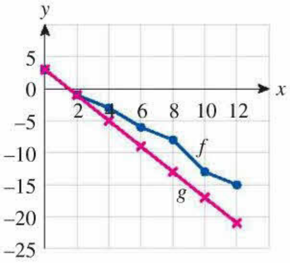


Figure 20

Finding a Linear Equation from Data

If we happen to know the slope and y-intercept of a line, writing down its equation is straightforward. For example, if we know that the slope is 3 and the y-intercept is −1, then the equation is $y = 3x - 1$. Sadly, the information we are given is seldom so convenient. For instance, we may know the slope and a point other than the y intercept, two points on the line, or other information. We therefore need to know how to use the information we are given to obtain the slope and the intercept.

Computing the Slope

We can always determine the slope of a line if we are given two (or more) points on the line, because any two points—say (x_1, y_1) and (x_2, y_2)—determine the line, and hence its slope. To compute the slope when given two points, recall the formula

$$\text{Slope} = m = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x}.$$

To find its slope, we need a run Δx and corresponding rise Δy. In Figure 21, we see that we can use $\Delta x = x_2 - x_1$, the change in the x-coordinate from the first point to the second, as our run, and $\Delta y = y_2 - y_1$, the change in the y-coordinate, as our rise. The resulting formula for computing the slope is given in the box.

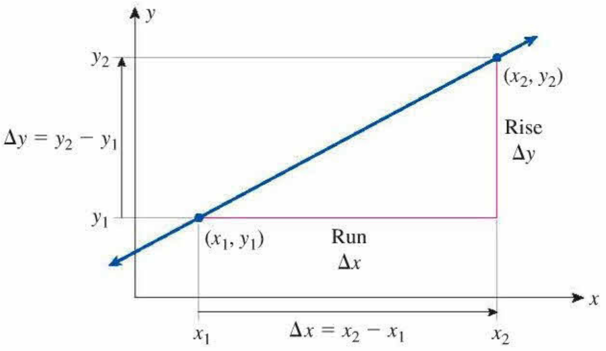


Figure 21

Computing the Slope of a Line

We can compute the slope m of the line through the points (x_1, y_1) and (x_2, y_2) by using

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Quick Examples

1. The slope of the line through $(x_1, y_1) = (1, 3)$ and $(x_2, y_2) = (5, 11)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11 - 3}{5 - 1} = \frac{8}{4} = 2.$$

Notice that we can use the points in the reverse order: If we take $(x_1, y_1) = (5, 11)$ and $(x_2, y_2) = (1, 3)$, we obtain the same answer:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 11}{1 - 5} = \frac{-8}{-4} = 2.$$

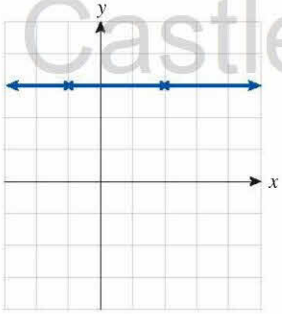
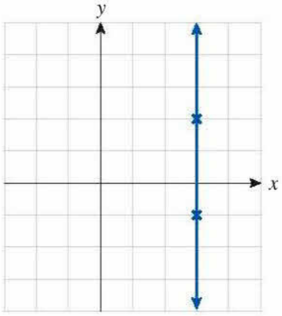


Figure 22



Vertical lines have undefined slope.

Figure 23

2. The slope of the line through $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (2, 1)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-2}{2-1} = \frac{-1}{1} = -1.$$

3. The slope of the line through $(2, 3)$ and $(-1, 3)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-3}{-1-2} = \frac{0}{-3} = 0.$$

A line of slope 0 has 0 rise, so is a *horizontal* line, as shown in Figure 22.

4. The line through $(3, 2)$ and $(3, -1)$ has slope

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-2}{3-3} = \frac{-3}{0},$$

which is undefined. The line passing through these points is *vertical*, as shown in Figure 23.

Computing the *y*-Intercept

Once we know the slope m of a line, and also the coordinates of a point (x_1, y_1), then we can calculate its intercept b as follows: The equation of the line must be

$$y = mx + b,$$

where b is as yet unknown. To determine b we use the fact that the line must pass through the point (x_1, y_1), so that (x_1, y_1) satisfies the equation $y = mx + b$. In other words,

$$y_1 = mx_1 + b.$$

Solving for b gives

$$b = y_1 - mx_1.$$

In summary:

Computing the *y*-Intercept of a Line

The y-intercept of the line passing through (x_1, y_1) with slope m is

$$b = y_1 - mx_1.$$

Quick Example

The line through $(2, 3)$ with slope 4 has

$$b = y_1 - mx_1 = 3 - (4)(2) = -5.$$

Its equation is therefore

$$y = mx + b = 4x - 5.$$

EXAMPLE 2 Finding Linear Equations

Find equations for the following straight lines.

a. Through the points (1, 2) and (3, −1)

b. Through (2, −2) and parallel to the line $3x + 4y = 5$

c. Horizontal and through (−9, 5)

d. Vertical and through (−9, 5)

Solution

a. To write down the equation of the line, we need the slope m and the y-intercept b.

- ***Slope*** Because we are given two points on the line, we can use the slope formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 2}{3 - 1} = -\frac{3}{2}$$

- ***Intercept*** We now have the slope of the line, $m = -3/2$, and also a point—we have two to choose from, so let us choose $(x_1, y_1) = (1, 2)$. We can now use the formula for the y-intercept:

$$b = y_1 - mx_1 = 2 - \left(-\frac{3}{2}\right)(1) = \frac{7}{2}$$

Thus, the equation of the line is

$$y = -\frac{3}{2}x + \frac{7}{2}. \qquad y = mx + b$$

using Technology

See the Technology Guides at the end of the chapter for detailed instructions on how to obtain the slope and intercept in Example 2(a) using a TI-83/84 Plus or Excel. Here is an outline:

TI-83/84 Plus

`STAT` EDIT; Enter values of x and y in lists L_1 and L_2.
Slope: Highlight the heading L_3 and enter
`"ΔList(L2)/ΔList(L1)"`
Intercept: Highlight the heading L_4 and enter
`"L2-sum(L3)*L1"` [More details on page 110.]

Excel

Enter headings x, y, m, b in cells A1–D1, and the values (x, y) in cells A2–B3. Enter
`=(B3-B2)/(A3-A2)`
in cell C2, and
`=B2-C2*A2`
in cell D2. [More details on page 116.]

b. Proceeding as before,

- ***Slope*** We are not given two points on the line, but we are given a parallel line. We use the fact that *parallel lines have the same slope*. (Why?) We can find the slope of $3x + 4y = 5$ by solving for y and then looking at the coefficient of x:

$$y = -\frac{3}{4}x + \frac{5}{4} \qquad \text{To find the slope, solve for } y.$$

so the slope is $-3/4$.

- ***Intercept*** We now have the slope of the line, $m = -3/4$, and also a point $(x_1, y_1) = (2, -2)$. We can now use the formula for the y-intercept:

$$b = y_1 - mx_1 = -2 - \left(-\frac{3}{4}\right)(2) = -\frac{1}{2}$$

Thus, the equation of the line is

$$y = -\frac{3}{4}x - \frac{1}{2}. \qquad y = mx + b$$

c. We are given a point: (−9, 5). Furthermore, we are told that the line is horizontal, which tells us that the slope is $m = 0$. Therefore, all that remains is the calculation of the y-intercept:

$$b = y_1 - mx_1 = 5 - (0)(-9) = 5$$

so the equation of the line is

$$y = 5.$$

$y = mx + b$

d. We are given a point: $(-9, 5)$. This time, we are told that the line is vertical, which means that the slope is undefined. Thus, we can't express the equation of the line in the form $y = mx + b$. (This formula makes sense only when the slope m of the line is defined.) What can we do? Well, here are some points on the desired line:

$$(-9, 1), (-9, 2), (-9, 3), \ldots,$$

so $x = -9$ and $y = $ *anything*. If we simply say that $x = -9$, then these points are all solutions, so the equation is $x = -9$.

Applications: Linear Models

Using linear functions to describe or approximate relationships in the real world is called **linear modeling**.

Recall from Section 1.2 that a **cost function** specifies the cost C as a function of the number of items x.

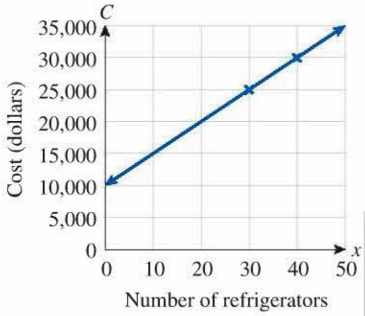


Figure 24

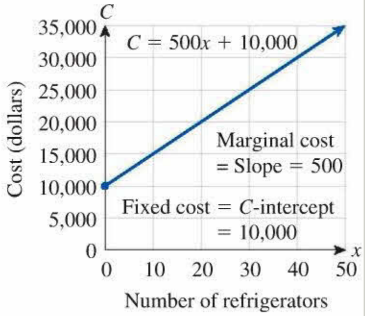


Figure 25

using Technology

To obtain the cost equation for Example 3 with technology, apply the Technology note for Example 2(a) to the given points (30, 25,000) and (40, 30,000) on the graph of the cost equation.

EXAMPLE 3 Linear Cost Function from Data

The manager of the FrozenAir Refrigerator factory notices that on Monday it cost the company a total of \$25,000 to build 30 refrigerators and on Tuesday it cost \$30,000 to build 40 refrigerators. Find a linear cost function based on this information. What is the daily fixed cost, and what is the marginal cost?

Solution We are seeking the cost C as a linear function of x, the number of refrigerators sold:

$$C = mx + b.$$

We are told that $C = 25{,}000$ when $x = 30$, and this amounts to being told that (30, 25,000) is a point on the graph of the cost function. Similarly, (40, 30,000) is another point on the line (Figure 24).

We can use the two points on the line to construct the linear cost equation:

- ***Slope*** $m = \dfrac{C_2 - C_1}{x_2 - x_1} = \dfrac{30{,}000 - 25{,}000}{40 - 30} = 500$ C plays the role of y.
- ***Intercept*** $b = C_1 - mx_1 = 25{,}000 - (500)(30) = 10{,}000$ We used the point $(x_1, C_1) = (30, 25{,}000)$.

The linear cost function is therefore

$$C(x) = 500x + 10{,}000.$$

Because $m = 500$ and $b = 10{,}000$ the factory's fixed cost is \$10,000 each day, and its marginal cost is \$500 per refrigerator. (See page 58 in Section 1.2.) These are illustrated in Figure 25.

Before we go on... Recall that, in general, the slope m measures the number of units of change in y per 1-unit change in x, so it is measured in units of y per unit of x:

$$\text{Units of Slope} = \text{Units of } y \text{ per unit of } x$$

In Example 3, y is the cost C, measured in dollars, and x is the number of items, measured in refrigerators. Hence,

$$\text{Units of Slope} = \text{Units of } y \text{ per Unit of } x = \text{Dollars per refrigerator}$$

The y-intercept b, being a value of y, is measured in the same units as y. In Example 3, b is measured in dollars. ■

In Section 1.2 we saw that a **demand function** specifies the demand q as a function of the price p per item, whereas a **supply function** specifies the supply q as a function of unit price p.

EXAMPLE 4 Linear Demand and Supply Functions from Data

You run a small supermarket, and must determine how much to charge for Hot'n'Spicy brand baked beans. The following chart shows weekly sales figures (the demand) for Hot'n'Spicy at two different prices, as well as the number of cans per week that you are prepared to place on sale (the supply) at these prices.

Price/Can	$0.50	$0.75
Demand (cans sold/week)	400	350
Supply (cans placed on sale/week)	300	500

a. Model these data with linear demand and supply functions. (See Example 4 in Section 1.2.)

b. How do we interpret the slope and q intercept of the demand equation? How do we interpret the slope of the supply equation?

c. Find the equilibrium price and graph demand and supply on the same set of axes. What happens if you charge more than the equilibrium price? What happens if you charge less?

Solution

a. Recall that a demand equation or demand function expresses demand q (in this case, the number of cans of beans sold per week) as a function of the unit price p (in this case, price per can). We model the demand using the two points we are given: (0.50, 400) and (0.75, 350).

$$\textbf{\textit{Slope:}}\ m = \frac{q_2 - q_1}{p_2 - p_1} = \frac{350 - 400}{0.75 - 0.50} = \frac{-50}{0.25} = -200$$

$$\textbf{\textit{Intercept:}}\ b = q_1 - mp_1 = 400 - (-200)(0.50) = 500$$

So, the demand equation is

$$q = -200p + 500. \qquad q = mp + b$$

To model the supply, we use the first and third rows of the table. We are again given two points: (0.50, 300) and (0.75, 500).

$$\textbf{\textit{Slope:}}\ m = \frac{q_2 - q_1}{p_2 - p_1} = \frac{500 - 300}{0.75 - 0.50} = \frac{200}{0.25} = 800$$

$$\textbf{\textit{Intercept:}}\ b = q_1 - mp_1 = 300 - (800)(0.50) = -100$$

So, the supply equation is

$$q = 800p - 100.$$

b. The key to interpreting the slope in the demand and supply equations is to recall (see the "Before we go on" note at the end of Example 3) that we measure the slope in *units of y per unit of x*. Let us consider the demand and supply equations separately:

Demand equation: Here, $m = -200$, and the units of m are units of q per unit of p, or the number of cans sold per dollar change in the price. Since m is negative, we see that the number of cans sold decreases as the price increases. We conclude that the weekly sales will drop by 200 cans per \$1-increase in the price.

To interpret the q intercept for the demand equation, recall that it gives the q-coordinate when $p = 0$. Hence, it is the number of cans the supermarket can "sell" every week if it were to give them away.*

*** NOTE** Does this seem realistic? Demand is not always unlimited if items were given away. For instance, campus newspapers are sometimes given away, and yet piles of them are often left untaken. Also see the "Before we go on..." discussion at the end of this example.

Supply equation: Here, $m = 800$, and the units of m are the number of cans you are prepared to supply per dollar change in the price. We conclude that the weekly supply will increase by 800 cans per \$1-increase in the price. (We do not interpret the q-intercept in the case of the supply equation; one cannot have a negative supply. See the "Before we go on" discussion at the end of the example.)

c. To find where the demand equals the supply, we equate the two functions:

$$\begin{aligned} \text{Demand} &= \text{Supply} \\ -200p + 500 &= 800p - 100 \\ -1000p &= -600, \end{aligned}$$

so

$$p = \frac{-600}{-1000} = \$0.60$$

This is the equilibrium price, as discussed in Example 4 of Section 1.2. We can find the corresponding demand by substituting 0.60 for p in the demand (or supply) equation.

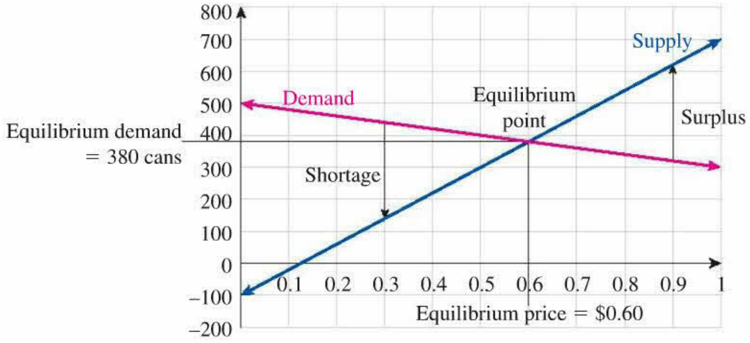
$$\text{Equilibrium demand} = -200(0.60) + 500 = 380 \text{ cans per week}$$

So, to balance supply and demand, you should charge \$0.60 per can of Hot'n'Spicy beans and you should place 380 cans on sale each week.

using Technology

To obtain the demand and supply equations for Example 4 with technology, apply the Technology note for Example 2(a) to the points (0.50, 400) and (0.75, 350) on the graph of the demand equation, and (0.50, 300) and (0.75, 500) on the graph of the supply equation.

➡ **Before we go on...** As we saw in Example 4 in Section 1.2, charging more than the equilibrium price will result in a surplus of Hot'n'Spicy beans, and charging less will result in a shortage. (See Figure 26.)



Equilibrium occurs at the point of intersection of the graphs of supply and demand.

Figure 26

Q: *Just how reliable are the linear models used in Example 4?*

A: The *actual* demand and supply graphs could in principle be obtained by tabulating demand and supply figures for a large number of different prices. If the resulting points were plotted on the *pq* plane, they would probably suggest curves and not a straight line. However, if you looked at a small enough portion of any curve, you could closely *approximate* it by a straight line. In other words, *over a small range of values of p, a linear model is accurate.* Linear models of real-world situations are generally reliable only for small ranges of the variables. (This point will come up again in some of the exercises.)

■

The next example illustrates modeling change over time t with a linear function of t.

EXAMPLE 5 Modeling Change Over Time: Growth of Sales

The worldwide market for portable navigation devices was expected to grow from 50 million units in 2007 to around 530 million units in 2015.*

a. Use this information to model annual worldwide sales of portable navigation devices as a linear function of time t in years since 2007. What is the significance of the slope?

b. Use the model to predict when annual sales of mobile navigation devices will reach 440 million units.

Solution

a. Since we are interested in worldwide sales s of portable navigation devices as a function of time, we take time t to be the independent coordinate (playing the role of x) and the annual sales s, in million of units, to be the dependent coordinate (in the role of y). Notice that 2007 corresponds to $t = 0$ and 2015 corresponds to $t = 8$, so we are given

---

* Sales were expected to grow to more than 500 million in 2015 according to a January 2008 press release by Telematics Research Group. Source: www.telematicsresearch.com.

the coordinates of two points on the graph of sales s as a function of time t: (0, 50) and (8, 530). We model the sales using these two points:

$$m = \frac{s_2 - s_1}{t_2 - t_1} = \frac{530 - 50}{8 - 0} = \frac{480}{8} = 60$$

$$b = s_1 - mt_1 = 50 - (60)(0) = 50$$

So, $s = 60t + 50$ million units. $s = mt + b$

The slope m is measured in units of s per unit of t; that is, millions of devices per year, and is thus the *rate of change of annual sales*. To say that $m = 60$ is to say that annual sales are increasing at a rate of 60 million devices per year.

b. Our model of annual sales as a function of time is

$$s = 60t + 50 \text{ million units.}$$

Annual sales of mobile portable devices will reach 440 million when $s = 440$, or

$$440 = 60t + 50$$

Solving for t, $60t = 440 - 50 = 390$

$$t = \frac{390}{60} = 6.5 \text{ years},$$

which is midway through 2013. Thus annual sales are expected to reach 440 million midway through 2013.

using Technology

To use technology to obtain s as a function of t in Example 5, apply the Technology note for Example 2(a) to the points (0, 50) and (8, 530) on its graph.

EXAMPLE 6 Velocity

You are driving down the Ohio Turnpike, watching the mileage markers to stay awake. Measuring time in hours after you see the 20-mile marker, you see the following markers each half hour:

Time (h)	0	0.5	1	1.5	2
Marker (mi)	20	47	74	101	128

Find your location s as a function of t, the number of hours you have been driving. (The number s is also called your **position** or **displacement**.)

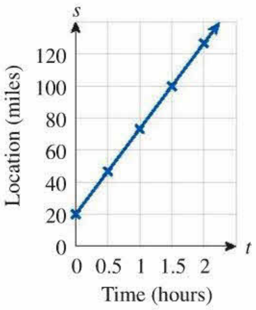


Figure 27

Solution If we plot the location s versus the time t, the five markers listed give us the graph in Figure 27.

These points appear to lie along a straight line. We can verify this by calculating how far you traveled in each half hour. In the first half hour, you traveled $47 - 20 = 27$ miles. In the second half hour you traveled $74 - 47 = 27$ miles also. In fact, you traveled exactly 27 miles each half hour. The points we plotted lie on a straight line that rises 27 units for every 0.5 unit we go to the right, for a slope of $27/0.5 = 54$.

To get the equation of that line, notice that we have the s-intercept, which is the starting marker of 20. Thus, the equation of s as a function of time t is

$$s(t) = 54t + 20.$$

We used s in place of y and t in place of x.

Notice the significance of the slope: For every hour you travel, you drive a distance of 54 miles. In other words, you are traveling at a constant velocity of 54 mph. We have uncovered a very important principle:

In the graph of displacement versus time, velocity is given by the slope.

using Technology

To use technology to obtain s as a function of t in Example 6, apply the Technology note for Example 2(a) to the points (0, 20) and (1, 74) on its graph.

Linear Change Over Time

If a quantity q is a linear function of time t,

$$q = mt + b,$$

then the slope m measures the **rate of change** of q, and b is the quantity at time $t = 0$, the **initial quantity**. If q represents the position of a moving object, then the rate of change is also called the **velocity**.

Units of *m* and *b*

The units of measurement of m are units of q per unit of time; for instance, if q is income in dollars and t is time in years, then the rate of change m is measured in dollars per year.

The units of b are units of q; for instance, if q is income in dollars and t is time in years, then b is measured in dollars.

Quick Example

If the accumulated revenue from sales of your video game software is given by $R = 2{,}000t + 500$ dollars, where t is time in years from now, then you have earned \$500 in revenue so far, and the accumulated revenue is increasing at a rate of \$2,000 per year.

Examples 3–6 share the following common theme.

General Linear Models

If $y = mx + b$ is a linear model of changing quantities x and y, then the slope m is the rate at which y is increasing per unit increase in x, and the y-intercept b is the value of y that corresponds to $x = 0$.

Units of *m* and *b*

The slope m is measured in units of y per unit of x, and the intercept b is measured in units of y.

Quick Example

If the number n of spectators at a soccer game is related to the number g of goals your team has scored so far by the equation $n = 20g + 4$, then you can expect 4 spectators if no goals have been scored and 20 additional spectators per additional goal scored.

FAQs

What to Use as *x* and *y*, and How to Interpret a Linear Model

Q: *In a problem where I must find a linear relationship between two quantities, which quantity do I use as x and which do I use as y?*

A: The key is to decide which of the two quantities is the independent variable, and which is the dependent variable. Then use the independent variable as x and the dependent variable as y. In other words, *y depends on x.*

Here are examples of phrases that convey this information, usually of the form *Find y [dependent variable] in terms of x [independent variable]:*

- Find the cost in terms of the number of items. y = Cost, x = # Items
- How does color depend on wavelength? y = Color, x = Wavelength

If no information is conveyed about which variable is intended to be independent, then you can use whichever is convenient.

Q: *How do I interpret a general linear model $y = mx + b$?*

A: The key to interpreting a linear model is to remember the units we use to measure m and b:

> *The slope m is measured in units of y per unit of x; the intercept b is measured in units of y.*

For instance, if $y = 4.3x + 8.1$ and you know that x is measured in feet and y in kilograms, then you can already say, "y is 8.1 kilograms when $x = 0$ feet, and increases at a rate of 4.3 kilograms per foot" without even knowing anything more about the situation!

1.3 EXERCISES

▼ more advanced ◆ challenging

T indicates exercises that should be solved using technology

In Exercises 1–6, a table of values for a linear function is given. Fill in the missing value and calculate m in each case.

1.

x	−1	0	1
y	5	8	

2.

x	−1	0	1
y	−1	−3	

3.

x	2	3	5
$f(x)$	−1	−2	

4.

x	2	4	5
$f(x)$	−1	−2	

5.

x	−2	0	2
$f(x)$	4		10

6.

x	0	3	6
$f(x)$	−1		−5

In Exercises 7–10, first find f(0), if not supplied, and then find the equation of the given linear function.

7.

x	−2	0	2	4
$f(x)$	−1	−2	−3	−4

8.

x	−6	−3	0	3
$f(x)$	1	2	3	4

9.

x	-4	-3	-2	-1
$f(x)$	-1	-2	-3	-4

10.

x	1	2	3	4
$f(x)$	4	6	8	10

In each of Exercises 11–14, decide which of the two given functions is linear and find its equation. HINT [See Example 1.]

11.

x	0	1	2	3	4
$f(x)$	6	10	14	18	22
$g(x)$	8	10	12	16	22

12.

x	-10	0	10	20	30
$f(x)$	-1.5	0	1.5	2.5	3.5
$g(x)$	-9	-4	1	6	11

13.

x	0	3	6	10	15
$f(x)$	0	3	5	7	9
$g(x)$	-1	5	11	19	29

14.

x	0	3	5	6	9
$f(x)$	2	6	9	12	15
$g(x)$	-1	8	14	17	26

In Exercises 15–24, find the slope of the given line, if it is defined.

15. $y = -\frac{3}{2}x - 4$

16. $y = \frac{2x}{3} + 4$

17. $y = \frac{x+1}{6}$

18. $y = -\frac{2x-1}{3}$

19. $3x + 1 = 0$

20. $8x - 2y = 1$

21. $3y + 1 = 0$

22. $2x + 3 = 0$

23. $4x + 3y = 7$

24. $2y + 3 = 0$

In Exercises 25–38, graph the given equation. HINT [See Quick Examples on page 77.]

25. $y = 2x - 1$

26. $y = x - 3$

27. $y = -\frac{2}{3}x + 2$

28. $y = -\frac{1}{2}x + 3$

29. $y + \frac{1}{4}x = -4$

30. $y - \frac{1}{4}x = -2$

31. $7x - 2y = 7$

32. $2x - 3y = 1$

33. $3x = 8$

34. $2x = -7$

35. $6y = 9$

36. $3y = 4$

37. $2x = 3y$

38. $3x = -2y$

In Exercises 39–54, calculate the slope, if defined, of the straight line through the given pair of points. Try to do as many as you can without writing anything down except the answer. HINT [See Quick Examples on page 79.]

39. $(0, 0)$ and $(1, 2)$

40. $(0, 0)$ and $(-1, 2)$

41. $(-1, -2)$ and $(0, 0)$

42. $(2, 1)$ and $(0, 0)$

43. $(4, 3)$ and $(5, 1)$

44. $(4, 3)$ and $(4, 1)$

45. $(1, -1)$ and $(1, -2)$

46. $(-2, 2)$ and $(-1, -1)$

47. $(2, 3.5)$ and $(4, 6.5)$

48. $(10, -3.5)$ and $(0, -1.5)$

49. $(300, 20.2)$ and $(400, 11.2)$

50. $(1, -20.2)$ and $(2, 3.2)$

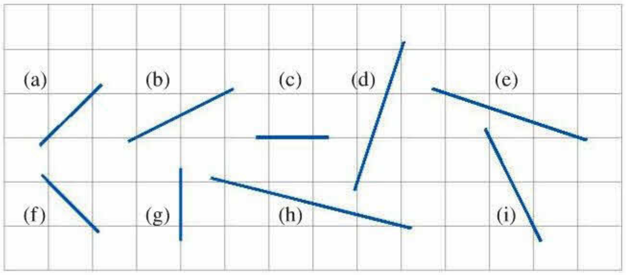
51. $(0, 1)$ and $\left(-\frac{1}{2}, \frac{3}{4}\right)$

52. $\left(\frac{1}{2}, 1\right)$ and $\left(-\frac{1}{2}, \frac{3}{4}\right)$

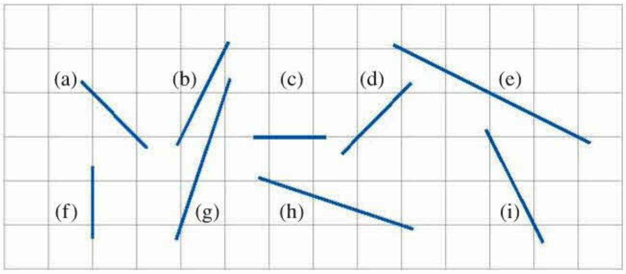
53. (a, b) and (c, d) $(a \neq c)$

54. (a, b) and (c, b) $(a \neq c)$

55. In the following figure, estimate the slopes of all line segments.



56. In the following figure, estimate the slopes of all line segments.



In Exercises 57–74, find a linear equation whose graph is the straight line with the given properties. HINT [See Example 2.]

57. Through $(1, 3)$ with slope 3

58. Through $(2, 1)$ with slope 2

59. Through $(1, -\frac{3}{4})$ with slope $\frac{1}{4}$

60. Through $(0, -\frac{1}{3})$ with slope $\frac{1}{3}$

61. Through $(20, -3.5)$ and increasing at a rate of 10 units of y per unit of x.

62. Through $(3.5, -10)$ and increasing at a rate of 1 unit of y per 2 units of x.

63. Through $(2, -4)$ and $(1, 1)$

64. Through $(1, -4)$ and $(-1, -1)$

65. Through $(1, -0.75)$ and $(0.5, 0.75)$

66. Through $(0.5, -0.75)$ and $(1, -3.75)$

67. Through $(6, 6)$ and parallel to the line $x + y = 4$

68. Through $(1/3, -1)$ and parallel to the line $3x - 4y = 8$

69. Through $(0.5, 5)$ and parallel to the line $4x - 2y = 11$

70. Through $(1/3, 0)$ and parallel to the line $6x - 2y = 11$

71. ▼ Through $(0, 0)$ and (p, q)

72. ▼ Through (p, q) and parallel to $y = rx + s$

73. ▼ Through (p, q) and (r, q) $(p \neq r)$

74. ▼ Through (p, q) and (r, s) $(p \neq r)$

APPLICATIONS

75. ***Cost*** The RideEm Bicycles factory can produce 100 bicycles in a day at a total cost of \$10,500 and it can produce 120 bicycles in a day at a total cost of \$11,000. What are the company's daily fixed costs, and what is the marginal cost per bicycle? HINT [See Example 3.]

76. ***Cost*** A soft-drink manufacturer can produce 1,000 cases of soda in a week at a total cost of \$6,000, and 1,500 cases of soda at a total cost of \$8,500. Find the manufacturer's weekly fixed costs and marginal cost per case of soda.

77. ***Cost: iPods*** It cost Apple approximately \$800 to manufacture 5 30-gigabyte video iPods and \$3,700 to manufacture 25.* Obtain the corresponding linear cost function. What was the cost to manufacture each additional iPod? Use the cost function to estimate the cost of manufacturing 100 iPods.

78. ***Cost: Xboxes*** If it costs Microsoft \$4,500 to manufacture 8 Xbox 360s and \$8,900 to manufacture 16,† obtain the corresponding linear cost function. What was the cost to manufacture each additional Xbox? Use the cost function to estimate the cost of manufacturing 50 Xboxes.

79. ***Demand*** Sales figures show that your company sold 1,960 pen sets each week when they were priced at \$1/pen set and 1,800 pen sets each week when they were priced at \$5/pen set. What is the linear demand function for your pen sets? HINT [See Example 4.]

80. ***Demand*** A large department store is prepared to buy 3,950 of your neon-colored shower curtains per month for \$5 each, but only 3,700 shower curtains per month for \$10 each. What is the linear demand function for your neon-colored shower curtains?

81. ***Demand for Cell Phones*** The following table shows worldwide sales of Nokia cell phones and their average wholesale prices in 2004:[27]

Quarter	***Second***	***Fourth***
Wholesale Price (\$)	111	105
Sales (millions)	45.4	51.4

a. Use the data to obtain a linear demand function for (Nokia) cell phones, and use your demand equation to predict sales if Nokia lowered the price further to \$103.

b. Fill in the blanks: For every ____ increase in price, sales of cell phones decrease by ___ units.

82. ***Demand for Cell Phones*** The following table shows projected worldwide sales of (all) cell phones and wholesale prices:[28]

Year	***2004***	***2008***
Wholesale Price (\$)	100	80
Sales (millions)	600	800

a. Use the data to obtain a linear demand function for cell phones, and use your demand equation to predict sales if the price was set at \$85.

b. Fill in the blanks: For every ____ increase in price, sales of cell phones decrease by ___ units.

83. ***Demand for Monorail Service, Las Vegas*** In 2005, the Las Vegas monorail charged \$3 per ride and had an average ridership of about 28,000 per day. In December, 2005 the Las Vegas Monorail Company raised the fare to \$5 per ride, and average ridership in 2006 plunged to around 19,000 per day.[29]

a. Use the given information to find a linear demand equation.

b. Give the units of measurement and interpretation of the slope.

c. What would be the effect on ridership of raising the fare to \$6 per ride?

---

*Source for cost data: Manufacturing & Technology News, July 31, 2007, Volume 14, No. 14, www.manufacturingnews.com.

†Source for estimate of marginal cost: www.isuppli.com.

[27] Source: Embedded.com/Companyreports December 2004.

[28] Wholesale price projections are the authors'. Source for sales prediction: I-Stat/NDR December, 2004.

[29] Source: *New York Times,* February 10, 2007, p. A9.

84. ***Demand for Monorail Service, Mars*** The Utarek monorail, which links the three urbynes (or districts) of Utarek, Mars, charged Ƶ5 per ride[30] and sold about 14 million rides per day. When the Utarek City Council lowered the fare to Ƶ3 per ride, the number of rides increased to 18 million per day.

a. Use the given information to find a linear demand equation.
b. Give the units of measurement and interpretation of the slope.
c. What would be the effect on ridership of raising the fare to Ƶ10 per ride?

85. ***Equilibrium Price*** You can sell 90 pet chias per week if they are marked at \$1 each, but only 30 each week if they are marked at \$2/chia. Your chia supplier is prepared to sell you 20 chias each week if they are marked at \$1/chia, and 100 each week if they are marked at \$2 per chia.

a. Write down the associated linear demand and supply functions.
b. At what price should the chias be marked so that there is neither a surplus nor a shortage of chias? HINT [See Example 4.]

86. ***Equilibrium Price*** The demand for your college newspaper is 2,000 copies each week if the paper is given away free of charge, and drops to 1,000 each week if the charge is 10¢/copy. However, the university is prepared to supply only 600 copies per week free of charge, but will supply 1,400 each week at 20¢ per copy.

a. Write down the associated linear demand and supply functions.
b. At what price should the college newspapers be sold so that there is neither a surplus nor a shortage of papers?

87. ***Pasta Imports*** During the period 1990–2001, U.S. imports of pasta increased from 290 million pounds in 1990 ($t = 0$) by an average of 40 million pounds/year.[31]

a. Use these data to express q, the annual U.S. imports of pasta (in millions of pounds), as a linear function of t, the number of years since 1990.
b. Use your model to estimate U.S. pasta imports in 2005, assuming the import trend continued.

88. ***Mercury Imports*** During the period 2210–2220, Martian imports of mercury (from the planet of that name) increased from 550 million kg in 2210 ($t = 0$) by an average of 60 million kg/year.

a. Use these data to express h, the annual Martian imports of mercury (in millions of kilograms), as a linear function of t, the number of years since 2010.
b. Use your model to estimate Martian mercury imports in 2230, assuming the import trend continued.

89. ***Satellite Radio Subscriptions*** The number of Sirius Satellite Radio subscribers grew from 0.3 million in 2003 to 3.2 million in 2005.[32]

a. Use this information to find a linear model for the number N of subscribers (in millions) as a function of time t in years since 2000.
b. Give the units of measurement and interpretation of the slope.
c. Use the model from part (a) to predict the 2006 figure. (The actual 2006 figure was approximately 6 million subscribers.)

90. ***Freon Production*** The production of ozone-layer damaging Freon 22 (chlorodifluoromethane) in developing countries rose from 200 tons in 2004 to a projected 590 tons in 2010.[33]

a. Use this information to find a linear model for the amount F of Freon 22 (in tons) as a function of time t in years since 2000.
b. Give the units of measurement and interpretation of the slope.
c. Use the model from part (a) to estimate the 2008 figure and compare it with the actual projection of 400 tons.

91. ***Velocity*** The position of a model train, in feet along a railroad track, is given by

$$s(t) = 2.5t + 10$$

after t seconds.

a. How fast is the train moving?
b. Where is the train after 4 seconds?
c. When will it be 25 feet along the track?

92. ***Velocity*** The height of a falling sheet of paper, in feet from the ground, is given by

$$s(t) = -1.8t + 9$$

after t seconds.

a. What is the velocity of the sheet of paper?
b. How high is it after 4 seconds?
c. When will it reach the ground?

93. ▼ ***Fast Cars*** A police car was traveling down Ocean Parkway in a high speed chase from Jones Beach. It was at Jones Beach at exactly 10 pm ($t = 10$) and was at Oak Beach, 13 miles from Jones Beach, at exactly 10:06 pm.

a. How fast was the police car traveling? HINT [See Example 6.]
b. How far was the police car from Jones Beach at time t?

---

[30] Ƶ designates Zonars, the official currency in Mars. See www.marsnext.com for details of the Mars colony, its commerce, and its culture.

[31] Data are rounded. Sources: Department of Commerce/*New York Times*, September 5, 1995, p. D4; International Trade Administration, March 31, 2002, www.ita.doc.gov/.

[32] Figures are approximate. Source: Sirius Satellite Radio/*New York Times,* February 20, 2008, p. A1.

[33] Figures are approximate. Source: Lampert Kuijpers (Panel of the Montreal Protocol), National Bureau of Statistics in China, via CEIC DSata/*New York Times,* February 23, 2007, p. C1.

94. ▼ ***Fast Cars*** The car that was being pursued by the police in Exercise 93 was at Jones Beach at exactly 9:54 pm ($t = 9.9$) and passed Oak Beach (13 miles from Jones Beach) at exactly 10:06 pm, where it was overtaken by the police.

a. How fast was the car traveling? HINT [See Example 6.]

b. How far was the car from Jones Beach at time t?

95. ▼ ***Fahrenheit and Celsius*** In the Fahrenheit temperature scale, water freezes at 32°F and boils at 212°F. In the Celsius scale, water freezes at 0°C and boils at 100°C. Assuming that the Fahrenheit temperature F and the Celsius temperature C are related by a linear equation, find F in terms of C. Use your equation to find the Fahrenheit temperatures corresponding to 30°C, 22°C, −10°C, and −14°C, to the nearest degree.

96. ▼ ***Fahrenheit and Celsius*** Use the information about Celsius and Fahrenheit given in Exercise 95 to obtain a linear equation for C in terms of F, and use your equation to find the Celsius temperatures corresponding to 104°F, 77°F, 14°F, and −40°F, to the nearest degree.

97. ▼ ***Income*** The well-known romance novelist Celestine A. Lafleur (a.k.a. Bertha Snodgrass) has decided to sell the screen rights to her latest book, *Henrietta's Heaving Heart*, to Boxoffice Success Productions for \$50,000. In addition, the contract ensures Ms. Lafleur royalties of 5% of the net profits.[34] Express her income I as a function of the net profit N, and determine the net profit necessary to bring her an income of \$100,000. What is her marginal income (share of each dollar of net profit)?

98. ▼ ***Income*** Due to the enormous success of the movie *Henrietta's Heaving Heart* based on a novel by Celestine A. Lafleur (see the Exercise 97), Boxoffice Success Productions decides to film the sequel, *Henrietta, Oh Henrietta.* At this point, Bertha Snodgrass (whose novels now top the best seller lists) feels she is in a position to demand \$100,000 for the screen rights and royalties of 8% of the net profits. Express her income I as a function of the net profit N and determine the net profit necessary to bring her an income of \$1,000,000. What is her marginal income (share of each dollar of net profit)?

99. ▼ ***Biology*** The Snowtree cricket behaves in a rather interesting way: The rate at which it chirps depends linearly on the temperature. One summer evening you hear a cricket chirping at a rate of 140 chirps/minute, and you notice that the temperature is 80°F. Later in the evening the cricket has slowed down to 120 chirps/minute, and you notice that the temperature has dropped to 75°F. Express the temperature T as a function of the cricket's rate of chirping r. What is the temperature if the cricket is chirping at a rate of 100 chirps/minute?

100. ▼ ***Muscle Recovery Time*** Most workout enthusiasts will tell you that muscle recovery time is about 48 hours. But it is not quite as simple as that; the recovery time ought to depend on the number of sets you do involving the muscle group in question. For example, if you do no sets of biceps exercises, then the recovery time for your biceps is (of course) zero. Let's assume that if you do three sets of exercises on a muscle group, then its recovery time is 48 hours. Use these data to write a linear function that gives the recovery time (in hours) in terms of the number of sets affecting a particular muscle. Use this model to calculate how long it would take your biceps to recover if you did 15 sets of curls. Comment on your answer with reference to the usefulness of a linear model.

101. ***Television Advertising*** The cost, in millions of dollars, of a 30-second television ad during the Super Bowl in the years 1990 to 2007 can be approximated by the following piecewise linear function ($t = 0$ represents 1990):[35]

$$C(t) = \begin{cases} 0.08t + 0.6 & \text{if } 0 \le t < 8 \\ 0.13t + 0.20 & \text{if } 8 \le t \le 17 \end{cases}$$

How fast and in what direction was the cost of an ad during the Super Bowl changing in 2006?

102. ***Processor Speeds*** The processor speed, in megahertz (MHz), of Intel processors could be approximated by the following function of time t in years since the start of 1995:[36]

$$P(t) = \begin{cases} 180t + 200 & \text{if } 0 \le t \le 5 \\ 3000t - 13{,}900 & \text{if } 5 < t \le 12 \end{cases}$$

How fast and in what direction was processor speed changing in 2005?

103. ▼ ***Investment in Gold*** Following are some approximate values of the Amex Gold BUGS Index.[37]

Year	1995	2000	2007
Index	200	50	470

Take t to be the year since 1995 and y to be the BUGS index.

a. Model the 1995 and 2000 data with a linear equation.

b. Model the 2000 and 2007 data with a linear equation.

c. Use the results of parts (a) and (b) to obtain a piecewise linear model of the gold BUGS index for 1995–2007.

d. Use your model to estimate the index in 2002.

104. ▼ ***Unemployment*** The following table shows the number of unemployed persons in the U.S. in 1994, 2000, and 2008.[38]

---

[34] Percentages of net profit are commonly called "monkey points." Few movies ever make a net profit on paper, and anyone with any clout in the business gets a share of the *gross*, not the net.

[35] Sources for data: *New York Times*, January 26, 2001, p. C1, http://money.cnn.com.

[36] Sources for data: Sandpile.org/*New York Times*, May 17, 2004, p. C1, www.Intel.com.

[37] BUGS stands for "basket of unhedged gold stocks." Figures are approximate. Sources: www.321gold.com, Bloomberg Financial Markets/*New York Times*, Sept 7, 2003, p. BU8, www.amex.com.

[38] Figures are seasonally adjusted and rounded. Source: U.S. Department of Labor, December, 2004, www.data.bls.gov.

Year	1994	2000	2008
Unemployment (Millions)	9	6	7

Take t to be the year since 1994 and y to be the number (in millions) of unemployed persons.

a. Model the 1994 and 2000 data with a linear equation.

b. Model the 2000 and 2008 data with a linear equation.

c. Use the results of parts (a) and (b) to obtain a piecewise linear model of the number (in millions) of unemployed persons for 1994–2008.

d. Use your model to estimate the number of unemployed persons in 2002.

105. ▼ ***Employment in Mexico*** The number of workers employed in manufacturing jobs in Mexico was 3 million in 1995, rose to 4.1 million in 2000, and then dropped to 3.5 million in 2004.[39] Model this number N as a piecewise-linear function of the time t in years since 1995, and use your model to estimate the number of manufacturing jobs in Mexico in 2002. (Take the units of N to be millions.)

106. ▼ ***Mortgage Delinquencies*** The percentage of borrowers in the highest risk category who were delinquent on their payments decreased from 9.7% in 2001 to 4.3% in 2004 and then shot up to 10.3% in 2007.[40] Model this percentage P as a piecewise-linear function of the time t in years since 2001, and use your model to estimate the percentage of delinquent borrowers in 2006.

COMMUNICATION AND REASONING EXERCISES

107. How would you test a table of values of x and y to see if it comes from a linear function?

108. You have ascertained that a table of values of x and y corresponds to a linear function. How do you find an equation for that linear function?

109. To what linear function of x does the linear equation $ax + by = c$ $(b \neq 0)$ correspond? Why did we specify $b \neq 0$?

110. Complete the following. The slope of the line with equation $y = mx + b$ is the number of units that ____ increases per unit increase in ____.

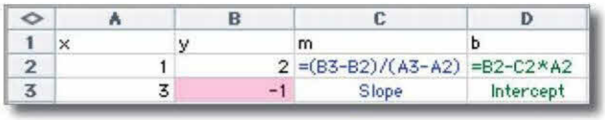
111. Complete the following. If, in a straight line, y is increasing three times as fast as x, then its ____ is ____.

112. Suppose that y is decreasing at a rate of 4 units per 3-unit increase of x. What can we say about the slope of the linear relationship between x and y? What can we say about the intercept?

113. If y and x are related by the linear expression $y = mx + b$, how will y change as x changes if m is positive? negative? zero?

114. Your friend April tells you that $y = f(x)$ has the property that, whenever x is changed by Δx, the corresponding change in y is $\Delta y = -\Delta x$. What can you tell her about f?

115. T ▼ Consider the following worksheet:



◇	A	B	C	D
1	x	y	m	b
2	1	2	=(B3-B2)/(A3-A2)	=B2-C2*A2
3	3	-1	Slope	Intercept

What is the effect on the slope of increasing the y-coordinate of the second point (the point whose coordinates are in Row 3)? Explain.

116. T ▼ Referring to the worksheet in Exercise 115, what is the effect on the slope of increasing the x-coordinate of the second point (the point whose coordinates are in row 3)? Explain.

117. If y is measured in bootlags[41] and x is measured in $\bar{\bar{\mathbf{Z}}}$ (zonars, the designated currency in Utarek, Mars)[42] and $y = mx + b$, then m is measured in _______ and b is measured in _______.

118. If the slope in a linear relationship is measured in miles per dollar, then the independent variable is measured in ______ and the dependent variable is measured in ______.

119. If a quantity is changing linearly with time, and it increases by 10 units in the first day, what can you say about its behavior in the third day?

120. The quantities Q and T are related by a linear equation of the form

$$Q = mT + b.$$

When $T = 0$, Q is positive, but decreases to a negative quantity when T is 10. What are the signs of m and b. Explain your answers.

121. ▼ The velocity of an object is given by $v = 0.1t + 20$ m/sec, where t is time in seconds. The object is

(A) moving with fixed speed
(B) accelerating
(C) decelerating
(D) impossible to say from the given information

122. ▼ The position of an object is given by $x = 0.2t - 4$, where t is time in seconds. The object is

(A) moving with fixed speed
(B) accelerating
(C) decelerating
(D) impossible to say from the given information

---

[39] Source: *New York Times*, February 18, 2007, p. WK4.

[40] The 2007 figure is projected from data through October 2006. Source: *New York Times*, February 18, 2007, p. BU9.

[41] An ancient Martian unit of length; one bootlag is the mean distance from a Martian's foreleg to its rearleg.

[42] Source: www.marsnext.com/comm/zonars.html.

123. ▼ Suppose the cost function is $C(x) = mx + b$ (with m and b positive), the revenue function is $R(x) = kx$ $(k > m)$ and the number of items is increased from the break-even quantity. Does this result in a loss, a profit, or is it impossible to say? Explain your answer.

124. ▼ You have been constructing a demand equation, and you obtained a (correct) expression of the form $p = mq + b$, whereas you would have preferred one of the form $q = mp + b$. Should you simply switch p and q in the answer, should you start again from scratch, using p in the role of x and q in the role of y, or should you solve your demand equation for q? Give reasons for your decision.

1.4 Linear Regression

We have seen how to find a linear model given two data points: We find the equation of the line that passes through them. However, we often have more than two data points, and they will rarely all lie on a single straight line, but may often come close to doing so. The problem is to find the line coming *closest* to passing through all of the points.

Suppose, for example, that we are conducting research for a company interested in expanding into Mexico. Of interest to us would be current and projected growth in that country's economy. The following table shows past and projected per capita gross domestic product (GDP)[43] of Mexico for 2000–2012.[44]

Year t ($t = 0$ represents 2000)	0	2	4	6	8	10	12	14
Per Capita GDP y ($1,000)	9	9	10	11	11	12	13	13

A plot of these data suggests a roughly linear growth of the GDP (Figure 28(a)).

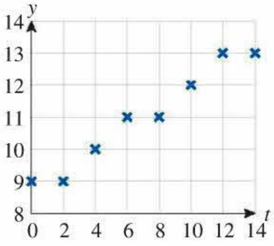


Figure 28(a)

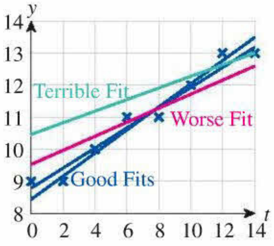


Figure 28(b)

These points suggest a roughly linear relationship between t and y, although they clearly do not all lie on a single straight line. Figure 28(b) shows the points together with several lines, some fitting better than others. Can we precisely measure which lines fit better than others? For instance, which of the two lines labeled as "good" fits in Figure 28(b) models the data more accurately? We begin by considering, for each value of t, the difference between the actual GDP (the **observed value**) and the GDP predicted by a linear equation (the **predicted value**). The difference between the predicted value and the observed value is called the **residual**.

$$\text{Residual} = \text{Observed Value} - \text{Predicted Value}$$

---

[43] The GDP is a measure of the total market value of all goods and services produced within a country.

[44] Data are approximate. Sources: CIA World Factbook/www.indexmundi.com, www.economist.com.