Regiomontanus and Trigonometry

The material presented in this teaching module is appropriate for an advanced high school or college trigonometry course.

1 The Early Years

The development of what we now call trigonometry is historically very closely linked to astronomy. From the time of the ancient Greeks and through the Middle Ages astronomers used trigonometric ratios to calculate the positions of stars, planets and other heavenly bodies. It is for this reason that most of the people we associate with the development of trigonometry were astronomers as well as mathematicians.

The foundations of modern trigonometry were laid sometime before 300 B.C.E., when the Babylonians divided the circle into 360 degrees. Within the next few centuries the Greeks adopted this measure of the circle, along with the further divisions of the circle into minutes and seconds. While attempting to explain the motions the planets, early astronomers found it necessary to solve for unknown sides and angles of triangles. The Greek astronomer Hipparchus (190-120 B.C.E) began a list of trigonometric ratios to aid in this endeavor. Hipparchus was also able to derive a half-angle formula for the sine [1].

The most influential book on ancient astronomy was the *Mathematiki Syntaxis (Mathematical Collection)*, otherwise known as *The Almagest*, by Claudius Ptolemy (c. 100-178 C.E.). This masterwork contains a complete description of the Greek model of the universe, and is considered by many to be the culmination of Greek astronomy [1]. It is interesting to note that *The Almagest* is the only completely comprehensive work on Greek astronomy to survive to the modern era. It contains 13 books dealing with everything from general assumptions of the science of the day, such as the size of the earth relative to the sphere of fixed stars, to the study of chords of a circle and the motions of the moon and all of planets known at the time. Only in part of book I of *The Almagest* does Ptolemy deal exclusively with what we would today consider to be trigonometry [3]. Section 10 of book I is called *On the Size of Chords in a Circle*, and in it is contained a famous table of chords, which today can be viewed as a table of sines for every angle up to 90º in quarter degree intervals [5].

Ptolemy's contribution to modern science and mathematics cannot be overstated, and in fact it wasn't until the 16th century that some of Ptolemy's ideas, such as the earth's position at the center of the universe, began to be questioned by the scientific community. His table of chords was used widely across Europe for hundreds of years without much improvement, until a man named Johannes Müller (Regiomontanus) published his *tabula fecunda* (fruitful table) in 1467 [2, intro.].

2 Along Comes Regiomontanus

Until the middle of the 15th century, works that contained methods in trigonometry were very closely tied to its practical use in astronomy. Around 1463 however, the first text dealing solely with trigonometry and triangles, called (appropriately enough) *De Triangulis Omnimodis* (*On Triangles of Every Kind*, or just *On Triangles*), was written by the astronomer Johannes Müller (1436-1476), otherwise known as Regiomontanus.

Regiomontanus was born on June 6th, 1436 in the town of Königsberg, Germany. His father was probably a miller, fairly well off, at least enough so to afford to send his son to Leipzig and Vienna to attend university. The young Regiomontanus showed extraordinary ability in both mathematics and astronomy at an early age, and by 12 was already enrolled in university at Leipzig, having found little challenge in the schools of his hometown of Königsberg. Also at the age of 12 (in 1448), Regiomontanus had calculated the positions of the planets for every day of the year, to a greater degree of accuracy than the Gutenberg calendar, which was first printed in the same year [4].

In 1450, Regiomontanus enrolled in the university in Vienna, which had a special reputation for its mathematics program. He received his baccalaureate in 1452, and master's in 1457. That same year he was named to the faculty of the university in Vienna. He stayed on as faculty until the death of his mentor, George Peuerbach, in 1461. He went to Italy, and it was in Venice in 1464 that he completed *On Triangles*. Regiomontanus was somewhat of a rare book aficionado, and during his stay in Italy he amassed quite a collection of rare works, either by copying them or buying them outright. Later in his life, he would become an avid publisher of, among other things, the works of ancient Greeks.

In the summer of 1467, after several years of traveling in Italy, Regiomontanus accepted the invitation of King Mathias of Hungary to become the librarian of the new Royal Library in Budapest. He was very well received there, partly because he was able to "foresee," using astrology, that the king would soon recover from a serious illness (which he did). Regiomontanus stayed in Hungary for the next four years, making astronomical observations and observational equipment for the king and the archbishop. Also during this time, Regiomontanus criticized a translation of *The Almagest* (among other things) done by George Trebizond, an act that would eventually cost him his life [2,4].

In 1471, Regiomontanus left Hungary for Nürnberg, a city which he had a particular affinity for. It was close to his hometown, and had become a center for academic activity. It was here that he did the majority of his publishing work. In his *Index*, he listed 41 various books on mathematics and astronomy that he intended to publish during his lifetime. He wasn't able to publish them all, and in 1475 the reigning Pope, Sixtus IV, summoned him to Rome to revise the Julian Calendar. He went reluctantly, and within a year he had died, poisoned by the sons of Trebizond [2].

Regiomontanus was highly praised in death, and in life was known as a genius of his time, a great astronomer and mathematician, and one who enjoyed bringing knowledge to the masses.

3 Regiomontanus on Triangles

In his book *On Triangles*, Regiomontanus collects the works of astronomers and mathematicians of previous eras (most notably Ptolemy) into one work on trigonometry. He provides clear examples and clarification when necessary, and in some cases improves on the methods of the ancients. He often cites Euclid's *The Elements*, and uses the axioms of that work extensively in his own.

In Theorem 20 of Book I, Regiomontanus states

> In every right triangle, one of whose acute vertices becomes the center of a circle and whose hypotenuse its radius, the side subtending this acute angle is the right sine* of the arc adjacent to that side and opposite the given angle, and the third side of the triangle is equal to the sine of the complement of the arc.

Let's examine this statement a little closer by following along with Regiomontanus' description of the theorem:

> If a right Δ *ABC* is given with *C* the right angle and *A* an acute angle, around the vertex of which a circle *BED* is described with the hypotenuse – that is, the side opposite the largest angle – as radius, and if side *AC* is extended sufficiently to meet the circumference of the circle at point *E*, then side *BC* opposite $\angle BAC$ is the sine of arc *BE* subtending the given angle, and furthermore the third side *AC* is equal to the right sine of the complement of arc *BE*.

(a) Using Regiomontanus' description and figure 1 below (a drawing very similar to this one was used by Regiomontanus in *On Triangles*), label the sine of arc *BE* (this could also be considered to be the sine of $\angle A$). What is the complement of arc *BE*? Label it, and label the (right) sine of the complement of arc *BE*. What do you think the modern term for the sine of the complement of *BE* might be?

* The term "right sine" is used to distinguish the sine from another part of the circle called the versed sine, which is no longer used (in figure 1, the versed sine is line segment EC). From here on, "right sine" and "sine" can be used interchangeably.

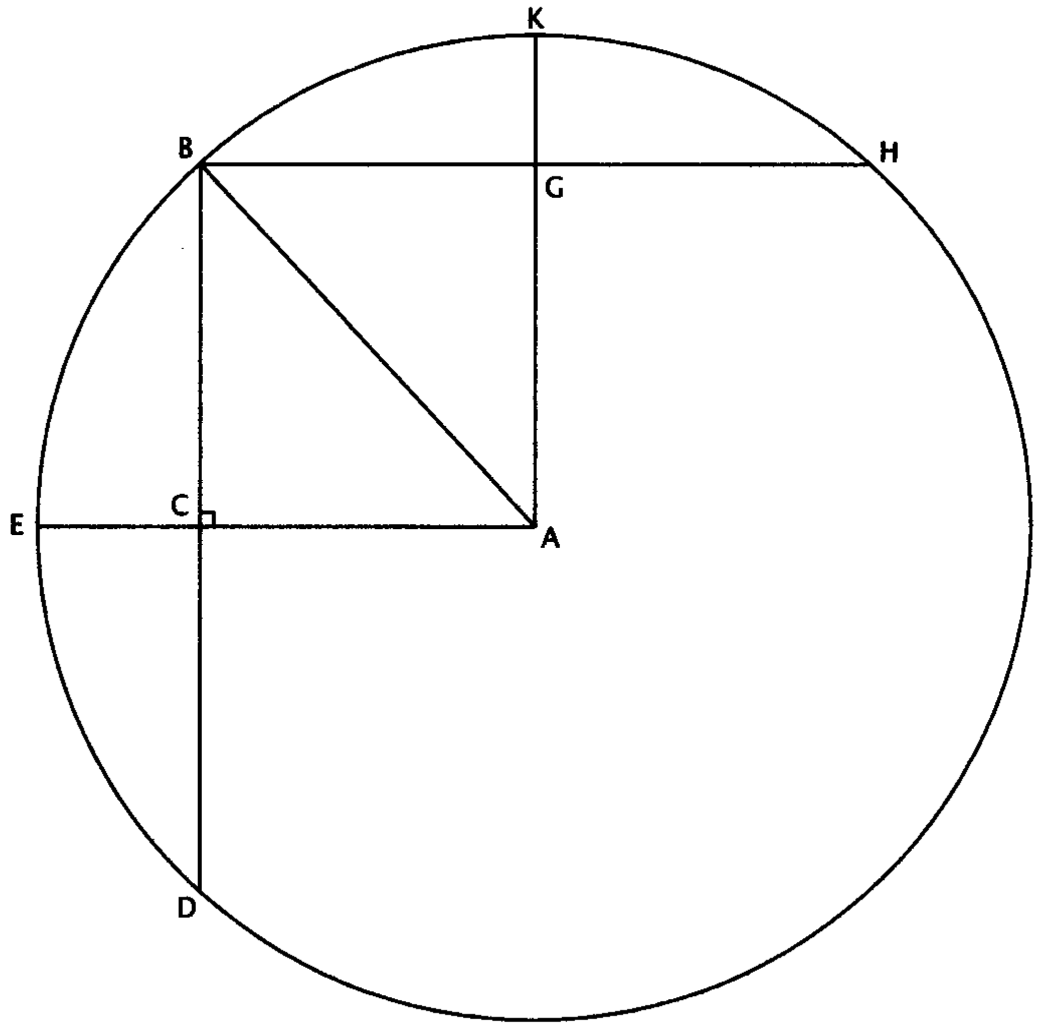

Figure 1.

Note that Regiomontanus refers to the sine of the *arc BE* rather than to the sine of $\angle A$. These two can be used interchangeably knowing that the length of the arc opposite an angle is equal to the product of the radius and the angle in radians. In modern trigonometry textbooks the sine function is called a circular function when it takes an arc length as an argument. Also note that in the above description, no mention is made of the length of side *AB* (the hypotenuse) of Δ *ABC*, which is also the radius of the circle. In modern trigonometry textbooks, we might see a geometric interpretation of the sine of an angle θ with a diagram similar to figure 1. In such a diagram, the sine of $\angle A$ would still be *BC*, only with the added condition that *BA*=1. Really the sine of an angle of a right triangle is the ratio of the length of the side opposite the angle to the hypotenuse of the triangle, so if the length of the hypotenuse is 1, then the sine of an angle is just the length of the side opposite the angle. Regiomontanus does not use this convention. In *On Triangles*, he uses a circle with radius 60,000 to formulate his table of sines (later, he actually made tables for which the radius of the circle was 6,000,000 and eventually

100,000,000) [6]. Figure 2 shows a simplified version of one of Regiomontanus' sine tables [2, derived]. Notice that the sine of 90º is 60,000. The sine of 90º is also called the *whole sine*, and it is the same as the radius of the circle.

Angle (degrees)	Sine	Angle	Sine	Angle	Sine
1	1047	31	30902	61	52477
2	2094	32	31795	62	52977
3	3140	33	32678	63	53460
4	4185	34	33552	64	53928
5	5229	35	34415	65	54378
6	6272	36	35267	66	54813
7	7312	37	36109	67	55230
8	8350	38	36940	68	55631
9	9386	39	37759	69	56015
10	10419	40	38567	70	56382
11	11449	41	39364	71	56731
12	12475	42	40148	72	57063
13	13497	43	40920	73	57378
14	14515	44	41680	74	57676
15	15529	45	42426	75	57956
16	16538	46	43160	76	58218
17	17542	47	43881	77	58462
18	18541	48	44589	78	58689
19	19534	49	45283	79	58898
20	20521	50	45963	80	59088
21	21502	51	46629	81	59261
22	22476	52	47281	82	59416
23	23444	53	47918	83	59553
24	24404	54	48541	84	59671
25	25357	55	49149	85	59772
26	26302	56	49742	86	59854
27	27239	57	50320	87	59918
28	28168	58	50883	88	59963
29	29089	59	51430	89	59991
30	30000	60	51962	90	60000

Figure 2.

(b) Compare some of the sines of the angles in figure 2 with modern values for the sines of those angles. How could we use Regiomontanus' table to arrive at the modern value for the sine of an (integer) angle?

To get a better idea of what a sine table is and where it comes from, let's compare the sine of a particular angle from figure 2 with the modern value of sine for that angle and see what's really happening. When we choose an angle, we can form a triangle out of it with one vertex on the circle and another at the center of the circle (as drawn in figure 1), and no matter which triangle is formed it will have a hypotenuse (whole sine) of 60,000 (in the case of a unit circle, the hypotenuse will be 1). If we choose 25º, we see that it has a sine of 25,357. What does this really mean? It means that the ratio of the side of the triangle opposite this angle to the hypotenuse of the triangle is $\frac{25357}{60000}$. If we use the unit circle, we must instead use the modern value for the sine of 25º, which according to the TI-89 calculator is about .42262. In the unit circle the hypotenuse is 1, so the ratio of the side opposite the 25º angle to the hypotenuse is $\frac{.42262}{1}$, or just .42262. What is important is that the *ratio* is the same no matter what value we use for the whole sine (notice $\frac{25357}{60000} \approx .42262$). What should be noticed here is that the triangle formed from a 25º angle in the unit circle is *similar* to the triangle formed from the 25º angle in Regiomontanus' circle with radius 60,000, and in fact, similar to a triangle formed from a 25º angle in a circle of *any* radius. This is where the sine table comes in handy. A sine table is based on whatever value is chosen for the whole sine. In Regiomontanus' table, for a given angle we know that a right triangle with a hypotenuse of 60,000 can be formed from that angle. We will then know the ratio of the side opposite that angle to 60,000. If we have a right triangle with the same angle but a different hypotenuse, we know that this triangle will be similar to the one with a hypotenuse of 60,000, and will therefore have the same ratio between its sides. This means that if we have a right triangle with a given angle and side, we can find the other sides by using the sine table. Further, if we are given two sides of a right triangle, it is possible to find all of the angles in the triangle using a sine table.

Let's look at a couple of the theorems from Book I of *On Triangles* and see if we can make sense of them. It may be helpful to refer to figure 1 while reading the theorems, noting that *A* and *B* are switched. In Theorem 27, Regiomontanus states:

> When two sides of a right triangle are known, all the angles can be found.
> If one of the given sides is opposite the right angle, that is sufficient; if not, however, we will find it, also, by the preceding theorem, for without it, it will not be possible to handle the theorem.

(c) What is the preceding theorem to which Regiomontanus is referring? (Keep in mind that he restates ancient and well-known theorems in his book).

> Thus if Δ *ABC* is given with *C* a right angle and sides *AB* and *AC* known, then all the angles can be found.
> When a circle is described with $\angle$ *B*, which the given side *AC* subtends, as center and side *BA* as radius, then, by Theorem 20 above,

> *AC* will be the sine of its adjacent arc, which is opposite the angle, *ABC*, that we seek.

Regiomontanus continues with what he calls "the mechanics," a description of how to find the angle *ABC* that we seek.

> The mechanics: Take the value of the side subtending the right angle as the first number, and take the value of the side opposite the desired angle for the second number, while the value of the whole sine is the third number. Then multiply the second by the third and divide the product by the first, for the sine of the arc opposite the desired angle will result.

(d) Using these instructions, write an equation for the "sine of the arc opposite the desired angle". Call the arc s, the "first number" r, the "second number" y, and remember that the "third number", the whole sine, is 60,000.

> From the table of sines you may determine that arc, whose value equals the desired angle. If you subtract this angle from the value of a right angle, the number that remains is the second acute angle.

(e) Using the table of sines in figure 2, find the two acute angles in a right triangle *ABC* if *AB*=20, *AC*=12 and *BC*=16. Since the sine table only lists sines for integer angles, find the angle that most closely matches the value you obtain for the sine of the angle.

(f) The whole sine to which Regiomontanus refers, if taken to be 1, could today be considered to be the radius of the unit circle. Why do you suppose Regiomontanus used a value of 60,000 instead of 1 for his "unit circle"? (Notice that there are no decimals in the table but each sine is listed to 4 or 5 significant figures).

The next theorem we look at explains that if a side and an angle of a right triangle are given, then all of the rest of the sides and angles can be found. As Regiomontanus explains in Theorem 29 of Book I:

> When one of the two acute angles and one side of a right triangle are known, all the angles and sides may be found.
> If in right Δ *ABC* with *C* the right angle, $\angle B$ is known together with any one side – say, *AC* – then all its angles and sides may be found.

Regiomontanus then goes on to give an example of how to do this:

> For instance, let $\angle ABC$ be given as 36º and side *AB* as 20 feet. Subtract 36 from 90 to leave 54º, the size of $\angle BAC$. Moreover, from the table of sines it is found that line *AC* is 35267 while *BC* is 48541, when *AB*, the whole sine, is 60000.

Here we can see how the similar triangles are working: for an angle of 36º, if the hypotenuse is 60,000 then the side opposite the 36º angle will be 35,267. We can now compare our similar triangle that has a hypotenuse of 20 to the reference triangle obtained from the sine table.

(g) Set up a ratio based on the similar triangles described above, that involves side *AC*. Solve for side *AC* using this ratio. Do the same for side *BC*. Do this before continuing with the rest of Regiomontanus' example. It may be helpful to draw a picture.

Regiomontanus continues:

> Therefore multiplying 35267 by 20 yields 705340, which, divided by 60000, leaves about 11 $\frac{45}{60}$. Thus side *AC* will have 11 feet and $\frac{45}{60}$ – that is, three-fourths of one foot. Similarly multiply 48541 by 20, giving 970820, which, divided by 60000, leaves about 16 feet and 11 min.[*], the length of side *BC*.

(h) We see that Regiomontanus has solved for the length of the remaining two sides. Did your answer from (g) agree with his?

4 Regiomontanus and the Law of Sines

In Book II of *On Triangles*, Regiomontanus introduces the law of sines in his own words (though it was known before his time). In Theorem I, he states:

> In every rectilinear[†] triangle the ratio of one side to another side is as that of the right sine of the angle opposite one of the sides to the right sine of the angle opposite the other side.
> As we said elsewhere, the sine of an angle is the sine of the arc subtending that angle. Moreover, these sines must be related through one and the same radius of the circle or through several equal radii. Thus, if Δ *ABG* is a rectilinear triangle, then the ratio of side *AB* to side *AG* is as that of the sine of $\angle$ *AGB* to the sine of $\angle$ *ABG*; similarly, that of side *AB* to *BG* is as that of the sine of $\angle$ *AGB* to the sine of $\angle$ *BAG*.

[*] In Regiomontanus' time, the sexagesimal system was more widely used than it is today. In it, 11 minutes is $\frac{11}{60} \approx .18$ feet, so 16 feet and 11 minutes is about 16.18 feet.

[†] As opposed to a spherical triangle (a triangle formed on the surface of a sphere), which Regiomontanus addresses later in the book.

(i) Write the law of sines as Regiomontanus explains it. Check that it is equivalent to the law of sines as stated in a modern trigonometry textbook.

In Theorem 4 of Book II, Regiomontanus states:

> If in any scalene triangle two angles are given individually with any one of its sides, the other sides are easily measured.
> If any two angles of Δ *ABG*, having three unequal sides, are given together with one of its sides – for example, *AB* – then the other two sides can be found.

In Theorem 5, he says:

> When two sides of a triangle are given together with the angle opposite one of them, the other angles and the third side may be determined.
> If such a Δ *ABG* has two sides *AB* and *AG* known along with $\angle$ *ABG*, then the other two angles and the third side may be found.

Modern trigonometry textbooks explain that there are two cases when the law of sines is applicable to solve a triangle (that is, find all the unknown sides and angles). When we know two sides of a triangle and an angle not between those sides, it is usually referred to as an SSA triangle*. When we know two angles and one side, it is usually referred to as an ASA triangle. Theorems 4 and 5 explain that it is possible to solve such triangles using the law of sines.

(j) Use the sine table in figure 2 to solve the following triangle: Δ *ABC*, where *AB*=8, *BC*=10 and $\angle$ *BAC*=36º. Since the sine table is only divided into 1º increments, find the angle which is closest to the sine that you obtain from your calculations. (*Hint*: You may need to find a way to take the sine of an angle greater than 90º using the table). Check your solutions with a calculator.

5 Possibilities for Further Study

Regiomontanus never directly refers to the cosine function, but he uses it implicitly when he refers to the sine of the complement of an arc. He also never refers to the tangent function, though Hughes states in his introduction to *On Triangles*, "It seems likely that Regiomontanus knew of the tangent function when he wrote his *Triangles*. Why he did not use it is another question." [2]. The origins and uses of the cosine and tangent functions would be a possible extension to this teaching module, as well as the researching the later introduction of the secant, cosecant and cotangent functions, which were of little use to the practical minded astronomers of the 15th century.

* Note that in this case of the law of sines (SSA) it may be possible to construct more than one triangle with the given information. With the numbers given in (j), only one triangle is possible.

References:

[1] Katz, V. J., *A History of Mathematics*, Addison-Wesley Education Publishers, Inc., 1998.

[2] Müller, Johann, *On Triangles*, translated by Barnabas Hughes, The University of Wisconsin Press, Madison, Wisconsin, 1967.

[3] Hutchins, R. M., (editor) *Ptolemy, Copernicus, Kepler*, William Benton, Chicago, 1952.

[4] Zinner, E., *Regiomontanus: His Life and Work*, translated by Ezra Brown, North-Holland, Amsterdam, 1990.

[5] *Ptolemy's Table of Chords*, Glenn Elert, 28 June 1994, http://hypertextbook.com/eworld/chords.shtml

[6] *The Garden of Archimedes: A Museum for Math*, http://web.math.unifi.it/archimede/archimede_NEW_inglese/trigonometria/trigonometria/trigo3.html

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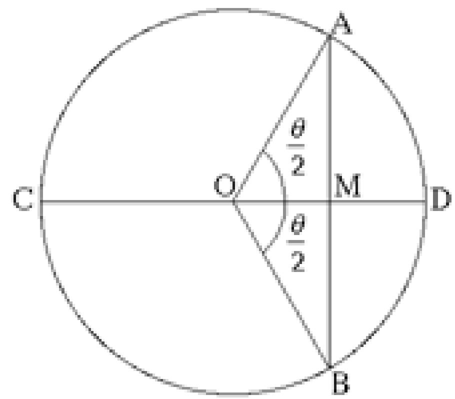

A Teaching Module

By Charles Mundy-Castle

Math 561

Spring, 2004