An Example of Induction: Fibonacci Numbers

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This short document is an example of an induction proof. Our goal is to rigorously prove something we observed experimentally in class, that every fifth Fibonacci number is a multiple of 5.

As usual in mathematics, we have to start by carefully defining the objects we are studying.

Definition. The sequence of **Fibonacci numbers**, $F_0, F_1, F_2, \ldots$, are defined by the following equations:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \end{aligned}$$

We now have to prove one of our early observations, expressing F_n as a sum of a multiple of 5, and a multiple of F_{n-5}.

Lemma 1. *If $n \geq 5$ is an integer, then*

$$F_n = 3F_{n-5} + 5F_{n-4}.$$

Proof. Repeatedly applying the recursion formula for Fibonacci numbers,

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} = (F_{n-2} + F_{n-3}) + F_{n-2} \\ &= 2F_{n-2} + F_{n-3} = 2(F_{n-3} + F_{n-4}) + F_{n-3} \\ &= 3F_{n-3} + 2F_{n-4} = 3(F_{n-4} + F_{n-5}) + 2F_{n-4} \\ &= 5F_{n-4} + 3F_{n-5}. \end{aligned}$$

□

Theorem 2. *The Fibonacci number F_{5k} is a multiple of 5, for all integers $m \geq 0$.*

Proof. Proof by induction on k. Since this is a proof by induction, we start with the base case of $k = 0$. That means, in this case, we need to compute $F_{5\times 0} = F_0$. But, by definition, $F_0 = 0 = 0 \times 5$, which is a multiple of 5.

Now comes the induction step, which is more involved. In the induction step, we assume the statement of our theorem is true for $k = m - 1$, and then prove that is true for $k = m$. So assume $F_{5(m-1)}$ is a multiple of 5, say

$$F_{5(m-1)} = 5x_m$$

for some integer x_m. [Strictly speaking, we don't need the subscript, but I am including it to match what we did in class.] We now need to show that F_{5m} is a multiple of 5. But

$$\begin{aligned} F_{5m} &= 3F_{5(m-1)} + 5F_{5m-4} && \text{by Lemma 1} \\ &= 3 \times 5x_m + 5F_{5m-4} && \text{by induction} \\ &= 5(3x_m + F_{5m-4}), && \text{by algebra} \end{aligned}$$

which is a multiple of 5 (since x_m and F_{5m-4} are integers).

We have shown that if $F_{5(m-1)}$ is a multiple of 5, then F_{5m} is also a multiple of 5. In other words, we have shown that if the statement of our theorem is true for $k = m - 1$, then the statement of our theorem is true for $k = m$. That means we have proved the induction step, and thus completed our proof. $\square$