This short document is an example of an induction proof. Our goal is to rigorously prove something we observed experimentally in class, that every fifth Fibonacci number is a multiple of 5.

As usual in mathematics, we have to start by carefully defining the objects we are studying.

Definition. The sequence of Fibonacci numbers, $F_0, F_1, F_2, \ldots$, are defined by the following equations:

\[
F_0 = 0 \\
F_1 = 1 \\
F_n + F_{n+1} = F_{n+2}
\]

Theorem 1. The Fibonacci number $F_{5k}$ is a multiple of 5, for all integers $k \geq 0$.

Proof. Proof by induction on $k$. Since this is a proof by induction, we start with the base case of $k = 0$. That means, in this case, we need to compute $F_{5\times0} = F_0$. But, by definition, $F_0 = 0 = 0 \times 5$, which is a multiple of 5.

Now comes the induction step, which is more involved. In the induction step, we assume the statement of our theorem is true for $k = n$, and then prove that is true for $k = n + 1$. So assume $F_{5n}$ is a multiple of 5, say

\[
F_{5n} = 5p
\]

for some integer $p$. We now need to show that $F_{5n+5} = F_{5(n+1)}$ is a multiple of 5. So we repeatedly use the recursive equation that defines Fibonacci
numbers, and algebra to compute

\[
F_{5n+5} = F_{5n+3} + F_{5n+4} = F_{5n+3} + (F_{5n+2} + F_{5n+3})
\]

\[
= F_{5n+2} + 2F_{5n+3} = F_{5n+2} + 2(F_{5n+1} + F_{5n+2})
\]

\[
= 2F_{5n+1} + 3F_{5n+2} = 2F_{5n+1} + 3(F_{5n} + F_{5n+1})
\]

\[
= 3F_{5n} + 5F_{5n+1}
\]

\[
= 3(5p) + 5F_{5n+1} \quad \text{by induction}
\]

\[
= 5(3p + F_{5n+1}) \quad \text{by algebra}
\]

which is a multiple of 5 (since \(F_{5n+1}\) and \(p\) are integers).

We have shown that if \(F_{5n}\) is a multiple of 5, then \(F_{5(n+1)}\) is also a multiple of 5. In other words, we have shown that if the statement of our theorem is true for \(k = n\), then the statement of our theorem is true for \(k = n + 1\). That means we have proved the induction step, and thus completed our proof. \(\square\)