Math 4326/5322 | **LINEAR ALGEBRA** | Spring 2019
Dr. Duval | **Main Exercises 5**

due 2pm, Thursday, March 7

1. Let $V = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 \colon x_1 + x_2 + x_3 = 0, x_4 + x_5 = 0\}$, and let $W = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 \colon x_1 + x_2 + x_4 = 0, x_2 + x_3 + x_5 = 0, x_1 + x_5 = 0\}$. Find a surjective map from V to W; be sure to *prove* that it is surjective. Can there be an injective map from V to W?

2. Let V and W be finite-dimensional vector spaces. Let U be a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

3. Let V and W be finite-dimensional vector spaces, and let $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

4. Let V and W be finite-dimensional vector spaces, and let $T \in \mathcal{L}(V, W)$. Prove that there exists a basis of V and a basis of W such that, with respect to these bases, all entries of $\mathcal{M}(T)$ are 0, except that the entries in row j, column j equal 1 for $1 \leq j \leq \dim \operatorname{range} T$.

5. (Graduate students only) Let V and W be finite-dimensional vector spaces, and let $T \in \mathcal{L}(V, W)$. Also let $v_1, \ldots, v_m$ be a basis of V. Prove that there exists a basis $w_1, \ldots, w_n$ of W such that, with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$, all the entries in the first column of $\mathcal{M}(T)$ are 0, except that the entry in the first row and first column *may* equal 1.

6. (Graduate students only) Let V and W be finite-dimensional vector spaces, and let $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} T_! \subseteq T_2$ if and only if there exists $S \in \mathcal{L}(V, V)$ such that $T_1 = T_2 S$.