Math 4326/5322 | **LINEAR ALGEBRA** | Spring 2019

Dr. Duval | **Main Exercises 8–corrected**

due 2pm, Thursday, April 11

1. Let V be a finite-dimensional complex vector space and let $T \in \mathcal{L}(V)$. Prove that if $1 \leq k \leq \dim V$, then T has an invariant subspace of dimension k.

2. Let V be a vector space, let $S, T \in \mathcal{L}(V)$, and assume that $ST = TS$. Prove that if $v \in V$ is an eigenvector for T with eigenvalue λ, then λ is also an eigenvalue for S. Find an eigenvector for λ with respect to S, and prove your answer is correct.

3. Now a sort of converse to the previous problem. Assume V is a finite-dimensional vector space, $\dim V = n$, and let $S, T \in \mathcal{L}(V)$. Prove that if T has n distinct eigenvalues, and S has the same eigenvectors as T, then $ST = TS$. (Note: S and T might have different eigen*values*.)

4. The Pell sequence $P_1, P_2, \ldots$ is defined by $P_1 = 1, P_2 = 2$, and

$$P_n = P_{n-2} + 2P_{n-1}$$

for $n \geq 3$. Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(x, y) = (y, x + 2y)$.

(a) Prove that $T^n(0, 1) = (P_n, P_{n+1})$ for every integer $n \geq 1$.

(b) Find the eigenvalues of T.

(c) Find a basis of $\mathbf{R}^2$ consisting of eigenvectors of T.

(d) Use the solution to part (c) to compute $T^n(0, 1)$. [Hint: Write $(0, 1)$ as a linear combination of eigenvectors.]

(e) Use your answers to parts (a) and (d) to prove that

$$P_n = \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}}$$

5. (Graduate students only) Let $R, T \in \mathcal{L}(\mathbf{F}^3)$, and assume R and T each have $3, 5, 9$ as eigenvalues. Prove that there exists an invertible operator $S \in \mathcal{L}(\mathbf{F}^3)$ such that $R = S^{-1}TS$.