A non-partitionable Cohen-Macaulay simplicial complex, and implications for Stanley depth

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Stanley: "I am glad that this problem has finally been put to rest, though I would have preferred a proof rather than a counterexample. Perhaps you can withdraw your paper from the arXiv and come up with a proof instead."

Stanley depth

Definition (Stanley)

Let $S = \Bbbk[x_1, \ldots, x_n]$, and let M be a $\mathbb{Z}^n$-graded S-module. Then sdepth M denotes the Stanley depth of M.

Conjecture (Stanley '82)

sdepth $M \geq$ depth M

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$\operatorname{sdepth} M \geq \operatorname{depth} M$

Theorem (Herzog, Jahan, Yassemi '08)

If I_Δ is the Stanley-Reisner ideal of a Cohen-Macaulay complex Δ, then the inequality $\operatorname{sdepth} S/I_\Delta \geq \operatorname{depth} S/I_\Delta$ is equivalent to the partitionability of Δ.

Corollary (DGKM '16)

Our counterexample disproves this conjecture as well.

Simplicial complexes

Definition (Simplicial complex)

Let V be set of vertices. Then Δ is a simplicial complex on V if:

- $\Delta \subseteq 2^V$; and
- if $\sigma \subseteq \tau \in \Delta$ implies $\tau \in \Delta$.

Higher-dimensional analogue of graph.

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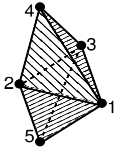
Definition (f-vector)

$f_i = f_i(\Delta) =$ number of i-dimensional faces of Δ. The f-vector of $(d-1)$-dimensional Δ is

$$f(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$$

Example

$f(\Delta) = (1, 5, 9, 6)$



Cohen-Macaulay complexes

Definition (Stanley-Reisner face-ring)

Assume Δ has vertices $1, \ldots, n$. Define $x_F = \prod_{j \in F} x_j$. Define I_Δ to be the ideal $I_\Delta = \langle x_F : F \notin \Delta \rangle$. The Stanley-Reisner face-ring is

$$\Bbbk[\Delta] = \Bbbk[x_1, \ldots, x_n]/I_\Delta.$$

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A ring R is Cohen-Macaulay when $\dim R = \operatorname{depth} R$.

In our setting $\dim \Bbbk[\Delta] = \dim \Bbbk[x_1, \ldots, x_n]/I_\Delta = d$.

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Remark (Munkres '84)

Being Cohen-Macaulay is topological, depends only on $|\Delta|$, geometric realization of Δ (and on the field $\mathbb{k}$).

h-vector

$$F(\Bbbk[\Delta], \lambda) = \sum_{\alpha \in \mathbb{Z}^n} \dim_{\Bbbk}(\Bbbk[\Delta]_\alpha)\mathbf{t}^\alpha$$

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This means

$$\sum_{i=0}^{d} f_{i-1} t^{d-i} = \sum_{k=0}^{d} h_k (t+1)^{d-k}.$$

The h-vector of Δ is $h(\Delta) = (h_0, h_1, \ldots, h_d)$. Coefficients not always non-negative

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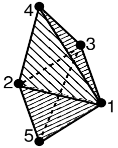
Example



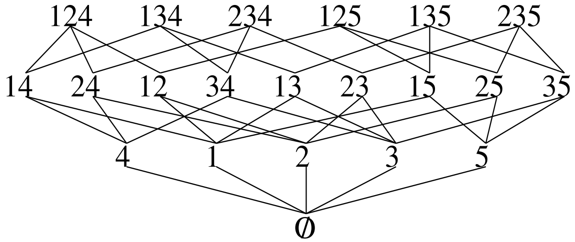
$f(\Delta) = (1, 5, 9, 6)$, and

$1t^3+5t^2+9t+6 = 1(t+1)^3+2(t+1)^2+2(t+1)^1+1$

so $h(\Delta) = (1, 2, 2, 1)$.

Partitionability

$$1t^3 + 5t^2 + 9t + 6 = {\color{red}1}(t+1)^3 + {\color{blue}2}(t+1)^2 + {\color{green}2}(t+1)^1 + 1$$



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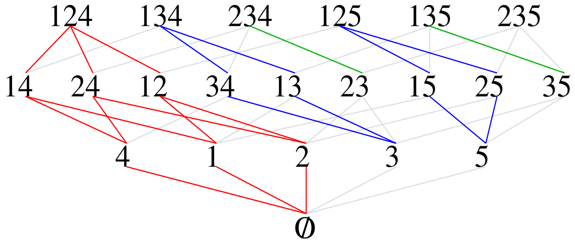

Definition (Partitionable)

When a simplicial complex can be partitioned like this, into Boolean intervals whose tops are facets, we say the complex is partitionable.

Key to construction: Relative complexes

Definition

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We found a relative CM complex, $Q_5 = (X_5, A_5)$ that is not partitionable. (Inside Ziegler's 3-dimensional non-shellable ball; $\dim X_5 = 3$ and X_5 has 5 facets.)

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If we glue together two copies of X_5 along A_5, is it partitionable?

Maybe. Some parts of A_5 might help partition one copy of X_5, while other parts of A_5 help partition the other copy of X_5.

Pigeonhole principle

Recall our example (X, A) is:

- relative Cohen-Macaulay
- not partitionable

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But the resulting complex is not actually a simplicial complex because of repeats.

Pigeonhole principle

Need our example (X, A) to be:

- relative Cohen-Macaulay
- not partitionable
- A vertex-induced (minimal faces of (X, A) are vertices)

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If we glue together many copies of X along A, at least one copy will be missing all of A! How many is enough? More than the number of all faces in A. Then the result will not be partitionable.

Remark

But the resulting complex is not actually a simplicial complex because of repeats. To avoid this problem, we need to make sure that A is vertex-induced. This means every face in X among vertices in A must be in A as well. (Minimal faces of (X, A) are vertices.)

Eureka!

By computer search, we found that if

- Z is Ziegler's non-shellable 3-ball, and
- $B = Z$ restricted to all vertices except 1,5,9 (B has 7 facets),

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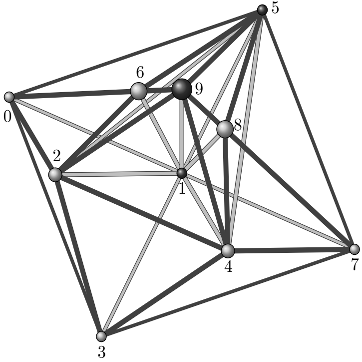
then $Q = (Z, B)$ satisfies all our criteria!
Also $Q = (X, A)$, where X has 14 facets, and A is 5 triangles:



1249	1269
1569	1589
1489	1458
1457	4578
1256	0125
0256	0123
1234	1347

Putting it all together

- Since A has 24 faces total (including the empty face), we know gluing together 25 copies of X along their common copy of A, the resulting (non-relative) complex C_{25} is CM, not partitionable.

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- In fact, computer search showed that gluing together only 3 copies of X will do it. Resulting complex C_3 has f-vector $(1, 16, 71, 98, 42)$.
- Later we found short proof by hand to show that C_3 works.

Stanley Decompositions

Definition

Let $S = \mathbb{k}[x_1, \ldots, x_n]$; $\mu \in S$ a monomial; and $A \subseteq \{x_1, \ldots, x_n\}$. The corresponding Stanley space in S is the vector space

$$\mu \cdot \mathbb{k}[A] \;=\; \mathbb{k}\text{-}\operatorname{span}\{\mu\nu \colon \operatorname{supp}(\nu) \subseteq A\}.$$

Let $I \subseteq S$ be a monomial ideal. A Stanley decomposition of S/I is a family of Stanley spaces

$$\mathcal{D} \;=\; \{\mu_1 \cdot \mathbb{k}[A_1],\; \ldots,\; \mu_r \cdot \mathbb{k}[A_r]\} \text{ such that}$$

$$S/I \;=\; \bigoplus_{i=1}^{r} \mu_i \cdot \mathbb{k}[A_i].$$

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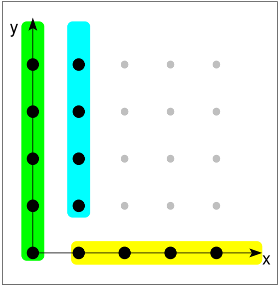
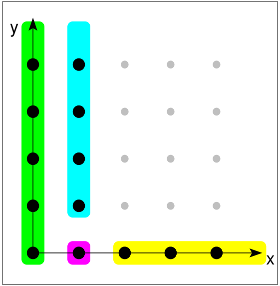
(And all of this works more generally for S-modules.)

Stanley Depth

Two Stanley decompositions of $R = \Bbbk[x,y]/\langle x^2 y \rangle$:





Definition

The Stanley depth of S/I is

$$\operatorname{sdepth} S/I \;=\; \max_{\mathcal{D}} \min\{|A_i|\}.$$

where $\mathcal{D}$ runs over all Stanley decompositions of S/I.

Depth Conjecture

Conjecture (Stanley '82)

For all monomial ideals I, $\operatorname{sdepth} S/I \geq \operatorname{depth} S/I$.

Theorem (Herzog, Jahan, Yassemi '08)

If I_Δ is the Stanley-Reisner ideal of a Cohen-Macaulay complex Δ, then the inequality $\operatorname{sdepth} S/I_\Delta \geq \operatorname{depth} S/I_\Delta$ *is equivalent to the partitionability of Δ.*

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Similarly, $\operatorname{sdepth} \Bbbk[Q_5] = 3$; $\operatorname{depth} \Bbbk[Q_5] = 4$. So that is a much smaller counterexample to the Depth Conjecture (for modules).