A non-partitionable Cohen-Macaulay simplicial complex

Art Duval[1], Bennet Goeckner[2], Caroline Klivans[3], Jeremy Martin[2]

[1]University of Texas at El Paso, [2]University of Kansas, [3]Brown University

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Cohen-Macaulay simplicial complexes

Definition (Stanley-Reisner face-ring)

Let Δ simplicial complex, vertices $1, \dots, n$. Define $x_F = \prod_{j \in F} x_j$.

$$\mathbb{k}[\Delta] := \mathbb{k}[x_1, \dots, x_n] / \langle x_F : F \notin \Delta \rangle.$$

Theorem (Reisner '76)

$\mathbb{k}[\Delta]$ *is Cohen-Macaulay (depth = dimension) if, for all* $\sigma \in \Delta$,

$$\tilde{H}_i(\operatorname{lk}_\Delta \sigma) = 0 \quad \textit{for } i < \dim \operatorname{lk}_\Delta \sigma.$$

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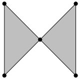
Remark (Munkres '84)

CM is topological; i.e., only depends on (the homeomorphism class of) the realization of Δ. In particular, spheres and balls are CM.

Example

 is not CM

h-vector

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$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{k=0}^{d} h_k t^{d-k}$$

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Equivalently,

$$\sum_{i=0}^{d} f_{i-1} t^{d-i} = \sum_{k=0}^{d} h_k (t+1)^{d-k}$$

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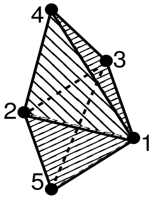

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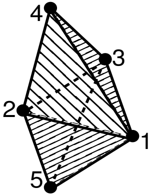

$f(\Delta) = (1, 5, 9, 6)$, and

$$1t^3 + 5t^2 + 9t + 6 = \color{red}{1}\color{black}(t+1)^3 + \color{blue}{2}\color{black}(t+1)^2 + \color{green}{2}\color{black}(t+1)^1 + 1$$

so $h(\Delta) = (\color{red}{1}\color{black}, \color{blue}{2}\color{black}, \color{green}{2}\color{black}, 1)$.

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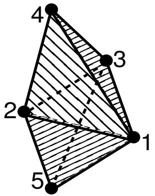

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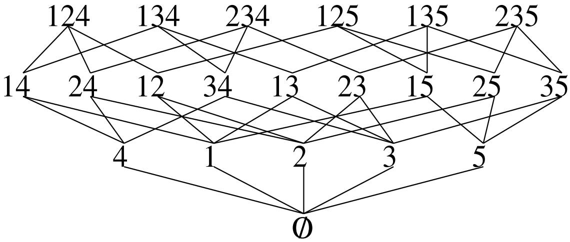
so $h(\Delta) = ({\color{red}1}, {\color{blue}2}, {\color{green}2}, 1)$.
Note that in this case, $h \geq 0$, because Δ is CM. But how could we see this combinatorially?

Partitionability

$$1t^3 + 5t^2 + 9t + 6 = 1(t+1)^3 + 2(t+1)^2 + 2(t+1)^1 + 1$$



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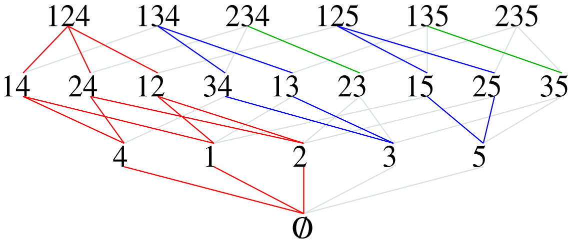

Definition (Partitionable)

When a simplicial complex can be partitioned like this, into Boolean intervals whose tops are facets, we say the complex is partitionable.

Relative complexes

Definition

If $\Gamma \subseteq \Delta$ are simplicial complexes, then (Δ, Γ) is a relative simplicial complex (this representation is not unique); think of Δ with Γ removed.

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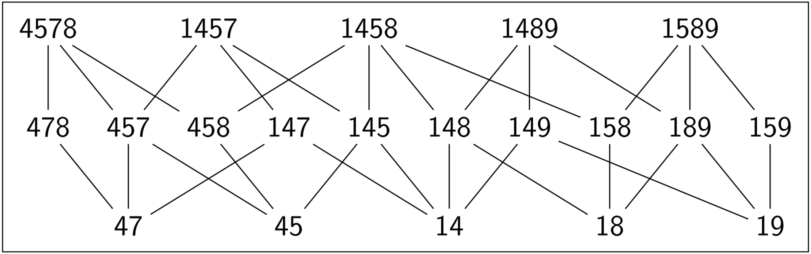
Remark

We found (inside Ziegler's 3-dimensional non-shellable ball) a relative CM complex $Q_5 = (X_5, A_5)$ that is not partitionable.



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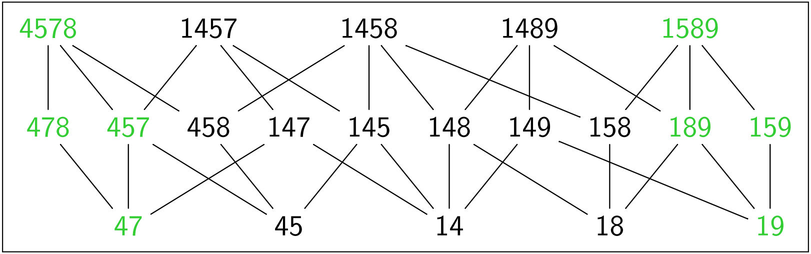

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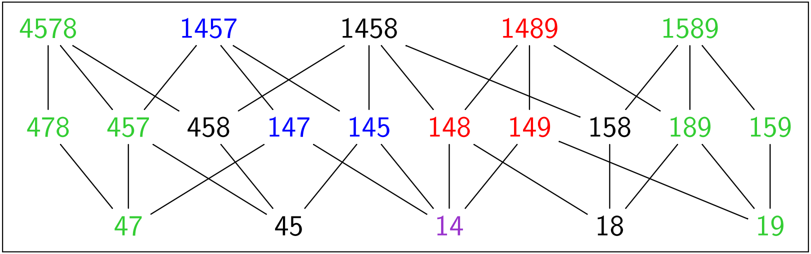

Gluing

Proposition

If X and (X, A) are CM and $\dim A = \dim X - 1$, then gluing together two copies of X along A gives a CM (non-relative) complex.

Question

If we glue together two copies of X along A, is it partitionable?

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If we glue together two copies of X along A, is it partitionable?

Maybe. Some parts of A can help partition one copy of X, other parts of A can help partition the other copy of X.

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Recall our example (X, A) is:

- relative Cohen-Macaulay
- not partitionable

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If we glue together many copies of X along A, at least one copy will be missing all of A!

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Pigeonhole principle

Need our example (X, A) to be:

- relative Cohen-Macaulay
- not partitionable
- A vertex-induced (minimal faces of (X, A) are vertices)

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Remark

But the resulting complex is not actually a simplicial complex because of repeats. To avoid this problem, we need to make sure that A is vertex-induced. This means every face in X among vertices in A must be in A as well. (Minimal faces of (X, A) are vertices.)

Eureka!

By computer search, we found that if

- Z is Ziegler's 3-ball, and
- $B = Z$ restricted to all vertices except 1,5,9 (B has 7 facets),

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By computer search, we found that if

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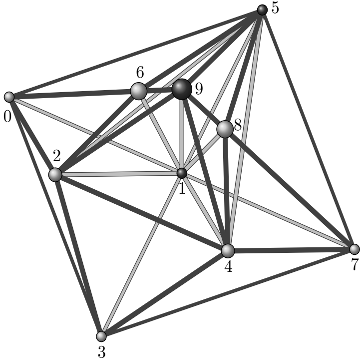
Also $Q = (X, A)$, where X has 14 facets, and A is 5 triangles:



1249	1269
1569	1589
1489	1458
1457	4578
1256	0125
0256	0123
1234	1347

Putting it all together

- Since A has 24 faces total (including the empty face), we know gluing together 25 copies of X along their common copy of A, the resulting (non-relative) complex C_{25} is CM, not partitionable.

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- In fact, computer search showed that gluing together only 3 copies of X will do it. Resulting complex C_3 has f-vector $(1, 16, 71, 98, 42)$.
- Later we found short proof by hand to show that C_3 works.

Stanley depth

Definition (Stanley)

If I is a monomial ideal in a polynomial ring S, then the Stanley depth $\operatorname{sdepth} S/I$ is a purely combinatorial analogue of depth, defined in terms of certain vector space decompositions of S/I.

Conjecture (Stanley '82)

For all monomial ideals I, $\operatorname{sdepth} S/I \geq \operatorname{depth} S/I$*.*

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Theorem (Herzog, Jahan, Yassemi '08)

Let $S = \Bbbk[x_1, \dots, x_n]$ and $I_\Delta = \langle x_F : F \notin \Delta \rangle$, so $\Bbbk[\Delta] = S/I_\Delta$.
If Δ is Cohen-Macaulay, then the inequality
$\operatorname{sdepth} S/I_\Delta \geq \operatorname{depth} S/I_\Delta$ *is equivalent to the partitionability of Δ.*

Corollary

Our counterexample disproves this conjecture as well.

Constructibility

Definition

A d-dimensional simplicial complex Δ is constructible if:

- it is a simplex; or
- $\Delta = \Delta_1 \cup \Delta_2$, where $\Delta_1, \Delta_2, \Delta_1 \cap \Delta_2$ are constructible of dimensions $d, d, d-1$, respectively.

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Corollary

Our counterexample is constructible, so the answer to this question is no.

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Is the partitionability conjecture true in 2 dimensions?

Save the conjecture: Strengthen the hypothesis

More open questions (based on what our counterexample is not):
Note that our counterexample is not a ball (3 balls sharing common 2-dimensional faces), but all balls are CM.

Question

Are simplicial balls partitionable?

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Question

Are simplicial balls partitionable?

Definition (Balanced)

A simplicial complex is balanced if vertices can be colored so that every facet has one vertex of each color.

Question

Are balanced Cohen-Macaulay complexes partitionable?

Save the conjecture: Weaken the conclusion

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What does the h-vector of a CM complex count?

Save the conjecture: Weaken the conclusion

Question

What *does* *the h-vector of a CM complex count?*

One possible answer (D.-Zhang '01) replaces Boolean intervals with "Boolean trees". But maybe there are other answers.