Cuts and flows in cell complexes

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CombinaTexas 2012
Southwestern University
April 22, 2012
Critical groups, cuts, and flows

Theorem (Bacher, de la Harpe, Nagnibeda)

\[ K(G) \cong \mathcal{C}^\# / \mathcal{C} \cong \mathcal{F}^\# / \mathcal{F} \cong \mathbb{Z}^{|E|} / (\mathcal{C} \oplus \mathcal{F}) \]

where \( G \) is a graph, \( K(G) \) is its critical group, \( \mathcal{C} \) is the cut lattice, and \( \mathcal{F} \) is the flow lattice.
Critical groups, cuts, and flows

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where $G$ is a graph, $K(G)$ is its critical group, $\mathcal{C}$ is the cut lattice, and $\mathcal{F}$ is the flow lattice.

Theorem (DKM)

$$0 \to \mathbb{Z}^n / (\mathcal{C} \oplus \mathcal{F}) \to K(\Sigma) \cong \mathcal{C}^\# / \mathcal{C} \to T(\tilde{H}_{d-1}(\Sigma, \mathbb{Z})) \to 0$$

$$0 \to T(\tilde{H}^d(\Sigma, \mathbb{Z})) \to \mathbb{Z}^n / (\mathcal{C} \oplus \mathcal{F}) \to K^*(\Sigma) \cong \mathcal{F}^\# / \mathcal{F} \to 0$$

where $\Sigma$ is a $d$-dimensional cell complex, $K(\Sigma)$ is its critical group, $K^*(\Sigma)$ is its cocritical group, $\mathcal{C}$ is the cut lattice, $\mathcal{F}$ is the flow lattice, and $T$ denotes torsion (finite) part of an abelian group.
Cuts and bonds

Let \( G \) be a connected graph.

**Definition**
A cut is a collection of edges in \( G \) whose removal disconnects the graph.

**Example**

![Diagram of a graph with a cut highlighted]
Cuts and bonds

Let $G$ be a connected graph

**Definition**

A *cut* is a collection of edges in $G$ whose removal disconnects the graph; a *bond* is a minimal cut.

**Example**

![Graph example](image-url)
Cuts and bonds

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Cuts and bonds

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**Example**

![Graph examples](image)

**Remark**
Using matroid language, bonds are cocircuits.
Cut space

The vertex star of every vertex is a cut;
Cut space

The vertex star of every vertex is a cut; it is also the coboundary of that vertex.

Definition

Cut space of $G$ is image of coboundary, $\text{im } \partial^*$, i.e., row-span of boundary [incidence] matrix.
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Example

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 \\
-1 & 1 & 0 & 0 & 1 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & -1 & -1
\end{pmatrix}
\]

Sum of first two rows ($\partial^*$ of north shore) is supported on bond.
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**Question**

What is a basis?
Fundamental bond

**Definition**

Given a spanning tree $T$

**Example**

![Diagram](image)
Fundamental bond

**Definition**
Given a spanning tree $T$ and an edge $e \in T$, the fundamental bond is the unique bond containing $e$, and no other edge from $T$.

**Example**

![Example Image]

Theorem
For a fixed spanning tree, the collection of fundamental bonds forms a basis of cut space.
Fundamental bond

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**Example**

![Example Diagram]

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Flows and circuits

**Definition**

A circuit is a closed path with no repeated vertices.
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Fundamental circuit

Definition
Given a spanning tree $T$ and an edge $e \not\in T$, the fundamental circuit is the unique circuit in $T \cup \{e\}$.

Example

- ![Diagram of a fundamental circuit](image)

Theorem
For a fixed spanning tree, the collection of fundamental circuits forms a basis of flow space.

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Cuts and flows in cell complexes
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**Example**

![Graphs](image1.png)

**Theorem**
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Cell complexes

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Cell complexes

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A cell complex $X$ is a finite CW-complex (i.e., collection of cells of different dimensions), with say $n$ facets and $p$ ridges, and a $p \times n$ cellular boundary matrix $\partial \in \mathbb{Z}^{p \times n}$.

Think the boundary of each facet being a $\mathbb{Z}$-linear combination of ridges.

Remark
Any $\mathbb{Z}$ matrix can be the boundary matrix of a cell complex.
Examples

2 3 0 0
0 0 5 7
0  2  2
1  0  0
−1  2  0
Cellular matroids

- Matroid whose elements are columns of boundary matrix
Cellular matroids

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- Dependent sets are the supports of the kernel of the boundary matrix
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- Bases?...
Spanning forests (Bolker; Kalai; DKM)

A Cellular spanning forest (CSF) is $\Upsilon \subset X$ such that:

$\Upsilon_{(d-1)} = X_{(d-1)}$ (same $(d-1)$-skeleton),
Spanning forests (Bolker; Kalai; DKM)

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- $\tilde{H}_d(\Upsilon; \mathbb{Q}) = 0$ and $\tilde{H}_{d-1}(\Upsilon; \mathbb{Q}) = \tilde{H}_{d-1}(X; \mathbb{Q})$
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Spanning forests (Bolker; Kalai; DKM)

A Cellular spanning forest (CSF) is \( \gamma \subset X \) such that:
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1. \( \tilde{H}_d(\gamma; \mathbb{Q}) = 0 \) and \( \tilde{H}_{d-1}(\gamma; \mathbb{Q}) = \tilde{H}_{d-1}(X; \mathbb{Q}) \)

2. Equivalently, \( \{ \partial F : F \in \gamma \} \) is a vector space basis for \( \text{im} \partial \)
Cut space and bonds

Definition

$i$-dimensional cut space of cell complex $X$ is

$$\text{Cut}_i(X) = \text{im}(\partial^*_i : C_{i-1}(X, \mathbb{R}) \to C_i(X, \mathbb{R})).$$

Remark

Cut space is the rowspace of the boundary matrix.
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A bond of $X$ is a minimal set of $i$-faces that support non-0 vector of $\text{Cut}_i(X)$

Remark
Cut space is the rowspace of the boundary matrix.

Remark
Bonds are the cocircuits of cellular matroid
Topological interpretation of bonds

Remark
Bonds are minimal for increasing \((i-1)\)-dimensional homology instead of decreasing \(i\)-dimensional homology

Examples
Characteristic vectors of bonds

Fix bond $B$

**Proposition**

\[ \text{Cut}_B(X) := (\{0\} \cup (\text{Cut}_i(X) \cap \{v : \text{supp}(v) = B\})) \text{ is 1-dimensional} \]

**Example**
Topological interpretation of characteristic vector

**Example**

If $B = \{F_5, F_7\}$, then $\text{Cut}_B$ spanned by $5F_5 + 7F_7$. 
Topological interpretation of characteristic vector

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If $B = \{F_5, F_7\}$, then $\text{Cut}_B$ spanned by $5F_5 + 7F_7$.

Theorem (DKM)

Let $A$ be a cellular spanning forest of $X/B$. Then $\text{Cut}_B(X)$ is spanned by

$$\chi(B, A) := \sum_{F \in B} \pm |\tilde{H}(A \cup F, \mathbb{Z})|F$$
Topological interpretation of characteristic vector

Example

If $B = \{F_5, F_7\}$, then $\chi(B, F_2) = 2(5F_5 + 7F_7)$, but $\chi(B, F_3) = 3(5F_5 + 7F_7)$.

Theorem (DKM)

Let $A$ be a cellular spanning forest of $X/B$. Then $\text{Cut}_B(X)$ is spanned by

$$\chi(B, A) := \sum_{F \in B} \pm |\tilde{H}(A \cup F, \mathbb{Z})|F$$

Definition

The characteristic vector of $B$ is $\chi(B, A)$
Fundamental bond

**Definition**
Given a spanning forest $\Upsilon$ and an face $F \in \Upsilon$, the **fundamental bond** is the unique bond containing $F$, and no other face from $\Upsilon$.

**Example**

\[
\begin{array}{c|c}
F & B \\
--- & --- \\
124 & \{124, 234\} \\
134 & \{124, 134\} \\
123 & \{234, 123, 125\} \\
135 & \{125, 135\} \\
235 & \{125, 235\}
\end{array}
\]
Fundamental bond

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**Example**

\[
\Upsilon = \{124, 134, 123, 135, 235\}
\]

<table>
<thead>
<tr>
<th>$F$</th>
<th>$B$</th>
</tr>
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<tbody>
<tr>
<td>124</td>
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<tr>
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<td>${234, 123, 125}$</td>
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**Theorem (DKM)**

*For a fixed spanning forest, the collection of characteristic vectors of fundamental bonds forms a basis of cut space.*
Flows and circuits

Definition

$i$-dimensional flow space of cell complex $X$ is

$$\text{Flow}_i(X) = \ker(\partial_i : C_{i-1}(X, \mathbb{R}) \to C_{i-1}(X, \mathbb{R})).$$
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A circuit of $X$ is a minimal set of $i$-faces that support non-0 vector of $\text{Flow}_i(X)$

Remark

Circuits are the circuits (minimal dependent sets) of cellular matroid.
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Example

Bipyramid
Characteristic vectors of circuits

Fix circuit $C$

**Proposition**

$Flow_C(X) := (\{0\} \cup (Flow_i(X) \cap \{v : \text{supp}(v) = C\}))$ is 1-dimensional

**Example**

Bipyramid
Topological interpretation of characteristic vector

Example

\[ \begin{array}{ccc}
2 & 2 & 1 \\
1 & 0 & -2 \\
-1 & 2 & 0
\end{array} \]

\[ \begin{array}{ccc}
0 & -2 & 2 \\
1 & 0 & -2 \\
-1 & 2 & 0
\end{array} \]

Theorem (DKM)

\[ \chi(C) = \sum_{F \in C} \pm |T \tilde{H}(C \setminus F, \mathbb{Z})| \]

Definition

The characteristic vector of \( C \) is \( \chi(C) \).
Topological interpretation of characteristic vector

Example

\[
\begin{pmatrix}
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\end{pmatrix}
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Theorem (DKM)

\[
\chi(C) = \sum_{F \in C} \pm |T\tilde{H}(C \setminus F, \mathbb{Z})|F
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spans \( \text{Cut}_C(X) \), where \( T \) stands for torsion part.
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\[ \tilde{H}(C \setminus F_1) = \mathbb{Z} \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2) ; \]

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\[\begin{bmatrix}
2 & 2 & 1 \\
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-1 & 2 & 0
\end{bmatrix}; \quad \tilde{H}(C \setminus F_1) = \mathbb{Z} \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2); \quad \chi(C) = (4, 2, 2)\]

Theorem (DKM)

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Definition

The characteristic vector of \( C \) is \( \chi(C) \)
**Fundamental circuit**

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$\Upsilon = \{124, 134, 123, 135, 235\}$

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Example
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\begin{align*}
\Upsilon &= \{124, 134, 123, 135, 235\} \\
F &\quad C \\
234 &\quad \{123, 124, 134, 234\} \\
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\end{align*}
\]

Theorem (DKM)
For a fixed spanning forest, the collection of characteristic vectors of fundamental circuits forms a basis of flow space.