Spanning Trees and Laplacians of Cubical Complexes

Art Duval[1] Caroline Klivans[2] Jeremy Martin[3]

[1]University of Texas at El Paso

[2]University of Chicago

[3]University of Kansas

CombinaTexas
University of Houston
April 25, 2009

Spanning Trees of Graph $G = (V, E)$

$T \subseteq E$ is a **spanning tree** of G when:

0. T contains all of V ($T_0 = V$)
1. connected ($\tilde{H}_0(T) = 0$)
2. no cycles ($\tilde{H}_1(T) = 0$)
3. $|T| = n - 1$

Note: If 0. holds, then any two of 1., 2., 3. together imply the third condition.

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$$D(G) = \text{diag}(\deg v_1, \ldots, \deg v_n)$$

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Defn 2: $L(G) = \partial(G)\partial(G)^T$

$$\partial(G) = \text{incidence matrix (boundary matrix)}$$

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Definition The reduced Laplacian matrix of graph G, denoted by $L_r(G)$.

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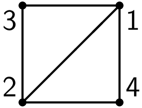
"Reduced": remove rows/columns corresponding to any one vertex

Example



$$\partial = \begin{array}{c|ccccc} & 12 & 13 & 14 & 23 & 24 \\ \hline 1 & -1 & -1 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & -1 & -1 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 & 0 & 1 \end{array}$$

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

Matrix-Tree Theorems

Version I Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then G has

$$\frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}$$

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Version II G has $|\det L_r(G)|$ spanning trees

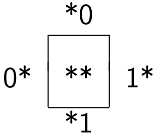
Cubical Complexes

Faces of Q_n, n-dimensional cube: $(0, 1, *)$-strings of length n. Dimension is number of *'s.

Vertices: $(0, 1)$-strings of length n

Edge in direction i: single * in position i.

Boundary: faces with one * converted to 0 or 1.



Cubical Complex: Subset of faces of Q_n such that if a face is included, then so is its boundary.

Spanning Trees

Let $\mathcal{Q}$ be a d-dimensional cubical complex.
$\Upsilon \subseteq \mathcal{Q}$ is a **cubical spanning tree** of $\mathcal{Q}$ when:

0. $\Upsilon_{(d-1)} = \mathcal{Q}_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $f_d(\Upsilon) = f_d(\mathcal{Q}) - \tilde{\beta}_d(\mathcal{Q}) + \tilde{\beta}_{d-1}(\mathcal{Q})$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.
- Works more generally for cellular complexes.

Example

The cubical biprism with equator, the boundary of $\langle$ ***0, **0* $\rangle$

This is the part where you look at the pretty ZomeTool model

- Let's count the spanning trees.

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- Let's count the spanning trees.
- $5 + 5$ spanning trees not containing face **00
- 5×5 spanning trees containing face **00
- 35 spanning trees total

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$\partial(\mathcal{Q})$ = signed boundary matrix

Example biprism

	00*0	01*0	0*00	0*10	10*0	00*0	11*0	...
0**0								
1**0								
*0*0								
*1*0								
**00								
**10								
...								

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Definition The reduced Laplacian matrix of d-dimensional cubical complex $\mathcal{Q}$, denoted by $L_r(\mathcal{Q})$.

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$\partial(\mathcal{Q}) =$ signed boundary matrix

"Reduced": remove rows/columns corresponding to spanning tree of $(d-1)$-dimensional faces

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*1*0								
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**10								
...								

Cubical Matrix-Tree Theorem — Version I

Theorem If $\mathcal{Q}$ a d-dimensional "metaconnected" cubical complex;
$(d-1)$-dimensional Laplacian $L_{d-1} = \partial_{d-1}\partial_{d-1}{}^T$;
s_d = product of nonzero eigenvalues of L_{d-1}, then

$$h_d := \sum_{\Upsilon \in CST(\mathcal{Q})} |\tilde{H}_{d-1}(\Upsilon)|^2 = \frac{s_d}{h_{d-1}} |\tilde{H}_{d-2}(\mathcal{Q})|^2$$

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Example Biprism: $h_d = \frac{(7^2 \cdot 5^4 \cdot 4 \cdot 3^2)(12)}{(7 \cdot 5^3 \cdot 4 \cdot 3^3 \cdot 2^2 \cdot 1)} = 35$

Cubical Matrix-Tree Theorem — Version II

- $\Gamma \in CST(\mathcal{Q}_{(d-1)})$
- ∂_Γ = restriction of ∂_d to faces not in Γ
- reduced Laplacian $L_\Gamma = \partial_\Gamma {\partial_\Gamma}^T$

Theorem [DKM]

$$h_d \;=\; \sum_{\Upsilon \in CST(\mathcal{Q})} |\tilde{H}_{d-1}(\Upsilon)|^2 \;=\; \frac{|\tilde{H}_{d-2}(\mathcal{Q};\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} |\det L_\Gamma|.$$

Note: The $|\tilde{H}_{d-2}|$ terms are often trivial.

Prisms

Definition If $\mathcal{Q}$ is a cubical complex, then $P\mathcal{Q}$, the prism over $\mathcal{Q}$ is the cubical complex

$$\{*, 0, 1\} \times \mathcal{Q}$$

in other words, for each string in $\mathcal{Q}$, make three new strings by putting *, 0, or 1 in front.

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Example (see the Zome Tools again!) Eigenvalues:

2 squares	5332100
prism	7554322

Eigenvalues of skeleta of cubes

Theorem The non-0 eigenvalues of the k-skeleton of Q_n are $2i$ with multiplicity $\binom{n}{i} \times \binom{i-1}{k-1}$ for $i = k, \ldots, n$
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Example 4-cube

k	eigenvalues
4	8^1
3	$8^3 6^4$
2	$8^3 6^8 4^6$
1	$8^1 6^4 4^6 2^4$
(0	2^4)

Shifted cubical complexes

Motivated by shifted simplicial complexes.

Given $\sigma \in Q_n = \{0, 1, *\}^n$, let $dir(\sigma) = \{i : \sigma_i = *\}$

A cubical complex $\mathcal{Q} \subseteq \{0, 1, *\}^n$ on n directions is **shifted** if:

1. If $\tau \in \mathcal{Q}$ and $dir(\sigma) < dir(\tau)$ (componentwise partial order), then $\sigma \in \mathcal{Q}$.

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***00	***01	***10	***11
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0***0	0***1	1***0	1***1
0**0*	0**1*	1**0*	1**1*

Near-Prisms

Definitions

- $del_{\mathcal{Q}}[i] := \{\sigma - \sigma_i : \sigma \in \mathcal{Q}, \sigma_i \neq *\}$
- $link_{\mathcal{Q}}[i] := \{\sigma - \sigma_i : \sigma \in \mathcal{Q}, \sigma_i = *\}$
- $\mathcal{Q}$ is a **near-prism** (in direction i) if
  - the boundary of $del[i]$ is contained in $link[i]$.
  - $0^i del[i] \cup 1^i del[i] \subseteq \mathcal{Q}$

Example Biprism is union of:

- prism over two open square (all faces using direction 1)
- four additional faces at ends (all faces not using direction 1)

Theorem (easy): A cubical complex is shifted iff it is a near-prism in direction 1, and its del[1] and link[1] are also shifted.

Laplacians

Theorem If $\mathcal{Q}$ is a near-prism in direction 1, then its (top-dimensional) Laplacian non-0 eigenvalues s are given by

$$s(\mathcal{Q}) = s(del[1]) \cup (2^{|link[1]|} + (s(del[1]) \cup s(link[1]))).$$

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Corollary Shifted cubical complexes are Laplacian integral.

Open Questions

- Shifted simplicial complexes have a nice formula for the Laplacian eigenvalues (transpose of degree sequence). Is there a formula for shifted cubical complexes?

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- The homology of shifted simplicial complexes (number of 0 eigenvalues) is easy to describe combinatorially. Can we do the same for shifted cubical complexes?
- Shifted simplicial complexes are extremal in several ways (including f-vectors, algebraic shifting). Are shifted cubical complexes extremal in any way?