Simplicial spanning trees

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Counting weighted spanning trees of $K_n$

**Theorem** [Cayley]: $K_n$ has $n^{n-2}$ spanning trees.

$T$ spanning tree: set of edges containing all vertices and

1. connected ($\tilde{H}_0(T) = 0$)
2. no cycles ($\tilde{H}_1(T) = 0$)
3. $|T| = n - 1$

Note: Any two conditions imply the third.
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- edges? No nice structure (can’t see $n^{n-2}$)
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$$\sum_{T \in \text{ST}(K_n)} \text{wt } T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2}$$
Example: $K_4$

- 4 trees like: $T = (x_1 x_2 x_3 x_4) x_2^2$
Example: $K_4$

- 4 trees like: $T = \begin{array}{c}
3 \\
2 \\
1
\end{array}$

- 12 trees like: $T = \begin{array}{c}
2 \\
3 \\
1 \\
4
\end{array}$

$\text{wt } T = (x_1x_2x_3x_4)x_2^2$

$\text{wt } T = (x_1x_2x_3x_4)x_1x_3$
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- 12 trees like: $T = \begin{array}{c}
2 \\
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\end{array}$

\[
\text{wt } T = (x_1 x_2 x_3 x_4) x_1 x_3
\]

Total is $(x_1 x_2 x_3 x_4)(x_1 + x_2 + x_3 + x_4)^2$. 

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Simplicial spanning trees
Laplacian

**Definition** The Laplacian matrix of graph $G$, denoted by $L(G)$. 

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"Reduced": remove rows/columns corresponding to any one vertex 

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Simplicial spanning trees
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Defn 1: $L(G) = D(G) - A(G)$

$D(G) = \text{diag}(\text{deg } v_1, \ldots, \text{deg } v_n)$

$A(G) = \text{adjacency matrix}$
Laplacian

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Defn 2: $L(G) = \partial(G)\partial(G)^T$

$\partial(G) = \text{incidence matrix (boundary matrix)}$
Laplacian

Definition The reduced Laplacian matrix of graph $G$, denoted by $L_r(G)$.

Defn 1: $L(G) = D(G) - A(G)$

$D(G) = \text{diag}(\text{deg } v_1, \ldots, \text{deg } v_n)$

$A(G) = \text{adjacency matrix}$

Defn 2: $L(G) = \partial(G)\partial(G)^T$

$\partial(G) = \text{incidence matrix (boundary matrix)}$

“Reduced”: remove rows/columns corresponding to any one vertex
Example

\[ \partial = \begin{pmatrix}
1 & -1 & -1 & -1 & 0 & 0 \\
2 & 1 & 0 & 0 & -1 & -1 \\
3 & 0 & 1 & 0 & 1 & 0 \\
4 & 0 & 0 & 1 & 0 & 1 \\
\end{pmatrix} \]

\[ L = \begin{pmatrix}
3 & -1 & -1 & -1 \\
-1 & 3 & -1 & -1 \\
-1 & -1 & 2 & 0 \\
-1 & -1 & 0 & 2 \\
\end{pmatrix} \]
Matrix-Tree Theorems

**Version I** Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of $L$. Then $G$ has

$$\lambda_1 \lambda_2 \cdots \lambda_{n-1}$$

$n$ spanning trees.

**Version II** $G$ has $\left| \det L_r(G) \right|$ spanning trees

**Proof** [Version II]

$$\det L_r(G) = \det \partial_r(G) \partial_r(G)^T = \sum_T (\det \partial_r(T))^2$$

$$= \sum_T (\pm 1)^2$$

by Binet-Cauchy
Example: $K_n$

$$L(K_n) = nl - J$$
$$L_r(K_n) = nl - J$$

$(n \times n);$

$(n-1 \times n-1)$
Example: $K_n$

$$L(K_n) = nl - J \quad (n \times n);$$

$$L_r(K_n) = nl - J \quad (n - 1 \times n - 1)$$

Version I: Eigenvalues of $L$ are $n - n$ (multiplicity 1), $n - 0$ (multiplicity $n - 1$), so

$$\frac{n^{n-1}}{n} = n^{n-2}$$
Example: $K_n$

\[
L(K_n) = nl - J \quad (n \times n);
\]
\[
L_r(K_n) = nl - J \quad (n - 1 \times n - 1)
\]

Version I: Eigenvalues of $L$ are $n - n$ (multiplicity 1), $n - 0$ (multiplicity $n - 1$), so

\[
\frac{n^{n-1}}{n} = n^{n-2}
\]

Version II:

\[
\det L_r = \prod \text{eigenvalues}
\]
\[
= (n - 0)^{(n-1)-1}(n - (n - 1))
\]
\[
= n^{n-2}
\]
Weighted Matrix-Tree Theorem

\[ \sum_{T \in ST(G)} \text{wt } T = | \det \hat{L}_r(G) |, \]

where \( \hat{L} \) is weighted Laplacian.
Defn 1: \( \hat{L}(G) = \hat{D}(G) - \hat{A}(G) \)
\[ \hat{D}(G) = \text{diag}(\hat{\deg} v_1, \ldots, \hat{\deg} v_n) \]
\[ \hat{\deg} v_i = \sum_{v_i v_j \in E} x_i x_j \]
\[ \hat{A}(G) = \text{adjacency matrix} \]
(entry \( x_i x_j \) for edge \( v_i v_j \))
Defn 2: \( \hat{L}(G) = \partial(G) B(G) \partial(G)^T \)
\[ \partial(G) = \text{incidence matrix} \]
\( B(G) \) diagonal, indexed by edges,
(entry \( \pm x_i x_j \) for edge \( v_i v_j \))
Example

\[ \hat{L} = \begin{pmatrix}
1(2 + 3 + 4) & -12 & -13 & -14 \\
-12 & 2(1 + 3 + 4) & -23 & -24 \\
-13 & -23 & 3(1 + 2) & 0 \\
-14 & -24 & 0 & 4(1 + 2)
\end{pmatrix} \]

\[ \det \hat{L}_r = (1234)(1 + 2)(1 + 2 + 3 + 4) \]
Threshold graphs: Order ideal definition

- Vertices $1, \ldots, n$

Example

```
\begin{tabular}{c c c c c}
1 & 2 & 3 & 4 & \\
2 & & & & \\
3 & & & & \\
4 & & & & \\
\end{tabular}
```
Threshold graphs: Order ideal definition

- Vertices 1, \ldots, n
- $E \in \mathcal{E}, i \not\in E, j \in E, i < j \Rightarrow E \cup i - j \in \mathcal{E}.$

Example

![Graph example]

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Simplicial spanning trees
Threshold graphs: Order ideal definition

- Vertices 1, \ldots, n
- \( E \in \mathcal{E}, i \notin E, j \in E, i < j \Rightarrow E \cup i - j \in \mathcal{E} \).
- Equivalently, the edges form an initial ideal in the componentwise partial order.

Example

\begin{itemize}
\item \begin{tikzpicture}
\node (1) at (0,0) {1};
\node (2) at (-1,-1) {2};
\node (3) at (1,-1) {3};
\node (4) at (0,-2) {4};
\draw (1) -- (2) -- (3) -- (4);\end{tikzpicture}
\item \begin{tikzpicture}
\node (1) at (0,0) {1};
\node (2) at (-1,1) {2};
\node (3) at (1,1) {3};
\node (4) at (2,0) {4};
\draw (1) -- (2) -- (3) -- (4);\end{tikzpicture}
\end{itemize}
Threshold graphs: Recursive building

Defn 2: Can build recursively, by adding isolated vertices, and coning.

\[ 3^* \]
Threshold graphs: Recursive building

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Threshold graphs: Recursive building

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Threshold graphs: Recursive building

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Eigenvalues of threshold graphs

**Theorem** [Merris ’94] Eigenvalues are given by the transpose of the Ferrers diagram of the degree sequence $d$.

\[ \text{Corollary} \quad \prod_{r \neq 1} (d^T)_r \text{ spanning trees} \]
Weighted spanning trees of threshold graphs

**Theorem** [Martin-Reiner ‘03; implied by Remmel-Williamson ‘02]: If $G$ is threshold, then

$$\sum_{T \in ST(G)} \text{wt} \ T = (x_1 \cdots x_n) \prod_{r \neq 1} \left( \sum_{i=1}^{d_T^r} x_i \right).$$

**Example**

$$\begin{pmatrix} 1234 \end{pmatrix}(1 + 2)(1 + 2 + 3 + 4)$$
Complete skeleta of simplicial complexes

Simplicial complex $\Sigma \subseteq 2^V$;
$F \subseteq G \in \Sigma \Rightarrow F \in \Sigma.$
Complete skeleta of simplicial complexes

Simplicial complex $\Sigma \subseteq 2^V$;
\[ F \subseteq G \in \Sigma \Rightarrow F \in \Sigma. \]

Complete skeleton The $k$-dimensional complete complex on $n$ vertices, i.e.,
\[ K^k_n = \{ F \subseteq V : |F| \leq k + 1 \} \]
(so $K_n = K^1_n$).

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Simplicial spanning trees
Simplicial spanning trees of $K_n^k$ [Kalai, ’83]

$\Upsilon \subseteq K_n^k$ is a simplicial spanning tree of $K_n^k$ when:

0. $\Upsilon_{(k-1)} = K_n^{k-1}$ (“spanning”);
1. $\tilde{H}_{k-1}(\Upsilon; \mathbb{Z})$ is a finite group (“connected”);
2. $\tilde{H}_k(\Upsilon; \mathbb{Z}) = 0$ (“acyclic”);
3. $|\Upsilon| = \binom{n-1}{k}$ (“count”).

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $k = 1$, coincides with usual definition.
Counting simplicial spanning trees of $K_n^k$

**Conjecture** [Bolker ’76]

$$\sum_{\Upsilon \in SST(K_n^k)} |\tilde{H}_{k-1}(\Upsilon)|^2 = n \binom{n-2}{k}$$
Counting simplicial spanning trees of $K^k_n$

**Theorem** [Kalai ’83]

$$\sum_{\mathcal{T} \in SST(K^k_n)} |\tilde{H}_{k-1}(\mathcal{T})|^2 = n^\binom{n-2}{k}$$
Weighted simplicial spanning trees of $K_n^k$

As before,

$$\text{wt }\Upsilon = \prod_{F \in \Upsilon} \text{wt } F = \prod_{F \in \Upsilon} \left( \prod_{v \in F} x_v \right)$$

Example:

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$

$$\text{wt }\Upsilon = x_1^5 x_2^4 x_3^3 x_4^3 x_5^3$$
Weighted simplicial spanning trees of $K_n^k$

As before,

$$\text{wt } \gamma = \prod_{F \in \gamma} \text{wt } F = \prod_{F \in \gamma} \left( \prod_{v \in F} x_v \right)$$

Example:

$$\gamma = \{123, 124, 125, 134, 135, 245\}$$

$$\text{wt } \gamma = x_1^5 x_2^4 x_3^3 x_4^3 x_5^3$$

**Theorem** [Kalai, '83]

$$\sum_{T \in \text{SST}(K_n)} |\tilde{H}_{k-1}(T)|^2 (\text{wt } T) = (x_1 \cdots x_n)^{\binom{n-2}{k-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{k}}$$

(Adin ('92) did something similar for complete $r$-partite complexes.)
Weighted simplicial spanning trees of $K_n^k$

As before,

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**Theorem** [Kalai, ’83]

$$\sum_{T \in \text{SST}(K_n)} |\tilde{H}_{k-1}(T)|^2 (\text{wt } T) = (x_1 \cdots x_n)^{(n-2)} (x_1 + \cdots + x_n)^{(n-k)}$$

(Adin ('92) did something similar for complete $r$-partite complexes.)
Proof uses determinant of reduced Laplacian of $K^n_k$. "Reduced" now means pick one vertex, and then remove rows/columns corresponding to all $(k - 1)$-dimensional faces containing that vertex.

$L = \partial \partial^T$

$\partial : \Delta_k \rightarrow \Delta_{k-1}$ boundary

$\partial^T : \Delta_{k-1} \rightarrow \Delta_k$ coboundary

Weighted version: Multiply column $F$ of $\partial$ by $x_F$
Example $n = 4, k = 2$

$$\partial^T = \begin{bmatrix}
12 & 13 & 14 & 23 & 24 & 34 \\
123 & -1 & 1 & 0 & -1 & 0 & 0 \\
124 & -1 & 0 & 1 & 0 & -1 & 0 \\
134 & 0 & -1 & 1 & 0 & 0 & -1 \\
234 & 0 & 0 & 0 & -1 & 1 & -1
\end{bmatrix}$$

$$L = \begin{pmatrix}
2 & -1 & -1 & 1 & 1 & 1 & 0 \\
-1 & 2 & -1 & -1 & 0 & 1 \\
-1 & -1 & 2 & 0 & -1 & -1 \\
1 & -1 & 0 & 2 & -1 & 1 \\
1 & 0 & -1 & -1 & 2 & -1 \\
0 & 1 & -1 & 1 & -1 & 2
\end{pmatrix}$$
Simplicial spanning trees of arbitrary simplicial complexes

Let $\Sigma$ be a $d$-dimensional simplicial complex. $\Upsilon \subseteq \Sigma$ is a **simplicial spanning tree** of $\Sigma$ when:

0. $\Upsilon_{(d-1)} = \Sigma_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $f_d(\Upsilon) = f_d(\Sigma) - \tilde{\beta}_d(\Sigma) + \tilde{\beta}_{d-1}(\Sigma)$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.
Example

Bipyramid with equator, $\langle 123, 124, 125, 134, 135, 234, 235 \rangle$

Let’s figure out all its simplicial spanning trees.
Denote by $\mathcal{T}(\Sigma)$ the set of simplicial spanning trees of $\Sigma$.

**Proposition** $\mathcal{T}(\Sigma) \neq \emptyset$ iff $\Sigma$ is **metaconnected**, i.e. (equivalently)

- homology type of wedge of spheres;
- $\tilde{H}_j(\Sigma; \mathbb{Z})$ is finite for all $j < \dim \Sigma$.

Many interesting complexes are metaconnected, including everything we'll talk about.
Simplicial Matrix-Tree Theorem — Version I

- $\Sigma$ a $d$-dimensional metaconnected simplicial complex
- $(d - 1)$-dimensional (up-down) Laplacian $L_{d-1} = \partial_{d-1}\partial^T_{d-1}$
- $s_d =$ product of nonzero eigenvalues of $L_{d-1}$.

**Theorem** [DKM]

$$h_d := \sum_{\gamma \in \mathcal{T}(\Sigma)} |\tilde{H}_{d-1}(\gamma)|^2 = \frac{s_d}{h_{d-1}} |\tilde{H}_{d-2}(\Sigma)|^2$$
Simplicial Matrix-Tree Theorem — Version II

- $\Gamma \in \mathcal{T}(\Sigma(d - 1))$
- $\partial_\Gamma = \text{restriction of } \partial_d \text{ to faces not in } \Gamma$
- reduced Laplacian $L_\Gamma = \partial_\Gamma \partial_\Gamma^*$

**Theorem [DKM]**

$$h_d = \sum_{\gamma \in \mathcal{T}(\Sigma)} |\tilde{H}_{d-1}(\gamma)|^2 = \frac{|\tilde{H}_{d-2}(\Sigma; \mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma; \mathbb{Z})|^2} \det L_\Gamma.$$  

**Note:** The $|\tilde{H}_{d-2}|$ terms are often trivial.
Weighted Simplicial Matrix-Tree Theorems

- Introduce an indeterminate $x_F$ for each face $F \in \Delta$
- Weighted boundary $\partial$: multiply column $F$ of (usual) $\partial$ by $x_F$
- $\partial_\Gamma = \text{restriction of } \partial_d \text{ to faces not in } \Gamma$
- Weighted reduced Laplacian $L = \partial_\Gamma \partial_\Gamma^*$

**Theorem** [DKM]

$$h_d := \sum_{\gamma \in \mathcal{T}(\Sigma)} |\tilde{H}_{d-1}(\gamma)|^2 \prod_{F \in \gamma} x_F^2 = \frac{s_d}{h_{d-1}} |\tilde{H}_{d-2}(\Sigma)|^2$$

$$h_d = \frac{|\tilde{H}_{d-2}(\Delta; \mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma; \mathbb{Z})|^2} \det L_\Gamma.$$
Definition of shifted complexes

- Vertices 1, \ldots, n
- \( F \in \Sigma, i \not\in F, j \in F, i < j \Rightarrow F \cup i - j \in \Sigma \)
- Equivalently, the \( k \)-faces form an initial ideal in the componentwise partial order.
- Example (bipyramid with equator)
  \( \langle 123, 124, 125, 134, 135, 234, 235 \rangle \)
Hasse diagram
Hasse diagram
Links and deletions

- **Deletion**, $\text{del}_1 \Sigma = \{ G : 1 \not\in G, G \in \Sigma \}$.
- **Link**, $\text{lk}_1 \Sigma = \{ F - 1 : 1 \in F, F \in \Sigma \}$.
- Deletion and link are each shifted, with vertices $2, \ldots, n$.
- **Example**:
  \[ \Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle \]
Links and deletions

- Deletion, $\text{del}_1 \Sigma = \{ G : 1 \notin G, G \in \Sigma \}$.
- Link, $\text{lk}_1 \Sigma = \{ F - 1 : 1 \in F, F \in \Sigma \}$.
- Deletion and link are each shifted, with vertices 2, \ldots, $n$.
- Example:

  \[
  \Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle \\
  \text{del}_1 \Sigma = \langle 234, 235 \rangle
  \]
Links and deletions

- Deletion, $\text{del}_1 \Sigma = \{ G : 1 \not\in G, G \in \Sigma \}$.
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- Example:

  \[ \Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle \]
  \[ \text{del}_1 \Sigma = \langle 234, 235 \rangle \]
  \[ \text{lk}_1 \Sigma = \langle 23, 24, 25, 34, 35 \rangle \]
Weighted spanning trees

- In Weighted Simplicial Matrix Theorem II, pick $\Gamma$ to be the set of all $(d - 1)$-dimensional faces containing vertex 1.
Weighted spanning trees

- In Weighted Simplicial Matrix Theorem II, pick $\Gamma$ to be the set of all $(d - 1)$-dimensional faces containing vertex 1.
- $H_{d-2}(\Gamma; \mathbb{Z})$ and $H_{d-2}(\Sigma; \mathbb{Z})$ are trivial, so,

$$h_d = \det L_\Gamma$$
Weighted spanning trees

In Weighted Simplicial Matrix Theorem II, pick $\Gamma$ to be the set of all $(d - 1)$-dimensional faces containing vertex 1.

$H_{d-2}(\Gamma; \mathbb{Z})$ and $H_{d-2}(\Sigma; \mathbb{Z})$ are trivial, so, by some easy linear algebra,

$$h_d = \det L_{\Gamma} = \left( \prod_{\sigma \in \text{lk}_1 \Sigma} X_{\sigma} \right) \det \left( X_1 I + L_{\text{del}_1 \Sigma, d-1} \right)$$

where $X_i = x_i^2$
Weighted spanning trees reduce to eigenvalues

- In Weighted Simplicial Matrix Theorem II, pick $\Gamma$ to be the set of all $(d - 1)$-dimensional faces containing vertex 1.

- $H_{d-2}(\Gamma; \mathbb{Z})$ and $H_{d-2}(\Sigma; \mathbb{Z})$ are trivial, so, by some easy linear algebra,

$$h_d = \det L_{\Gamma} = \left( \prod_{\sigma \in \text{lk}_1 \Sigma} X_{\sigma} \right) \det(X_1 I + L_{\text{del}_1 \Sigma, d-1})$$

$$= \left( \prod_{\sigma \in \text{lk}_1 \Sigma} X_{\sigma} \right) \left( \prod_{\lambda} \text{eval of } L_{\text{del}_1 \Sigma, d-1} \right) X_1 + \lambda,$$

where $X_i = x_i^2$
Eigenvalues

**Theorem** [D-Reiner, ’02]

Non-zero eigenvalues are given by the transpose of the Ferrers diagram of the (generalized) degree sequence $d$.

**Example**

```
2 3
5

2
3
4
5
```
Weighted Eigenvalues

**Theorem** [DKM]
Non-zero weighted eigenvalues are given by the transpose of the Ferrers diagram of the (generalized) degree sequence $d$.

**Example**

\[(2 + 3)(2 + 3 + 4 + 5)\]
Weighted enumeration of SST’s in shifted complexes

**Theorem** Let $\Lambda = \text{lk}_1 \Sigma$, $\Delta = \text{del}_1 \Sigma$,

**Example** bipyramid $\Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle$ again

$\Lambda = \text{lk}_1 \Sigma = \langle 23, 24, 25, 34, 35 \rangle$

$\Delta = \text{del}_1 \Sigma = \langle 234, 235 \rangle$
Weighted enumeration of SST’s in shifted complexes

**Theorem** Let \( \Lambda = \text{lk}_1 \Sigma \), \( \Delta = \text{del}_1 \Sigma \),

\[
h_2 = (23)(24)(25)(34)(35)(1 + (2 + 3))(1 + (2 + 3 + 4 + 5))111
\]

**Example** bipyramid \( \Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle \) again

\( \Lambda = \text{lk}_1 \Sigma = \langle 23, 24, 25, 34, 35 \rangle \)

\( \Delta = \text{del}_1 \Sigma = \langle 234, 235 \rangle \)
Weighted enumeration of SST’s in shifted complexes

Theorem Let $\Lambda = \text{lk}_1 \Sigma$, $\Delta = \text{del}_1 \Sigma$,

$$h_d = \prod_{\sigma \in \Lambda} X_{\sigma \cup 1} \prod_r \left( \sum_{i=1}^{1+(d(\Delta)^T)_r} X_i / X_1 \right)$$

Example bipyramid $\Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle$ again

$\Lambda = \text{lk}_1 \Sigma = \langle 23, 24, 25, 34, 35 \rangle$

$\Delta = \text{del}_1 \Sigma = \langle 234, 235 \rangle$

$$h_2 = (23)(24)(25)(34)(35)(1 + (2 + 3))(1 + (2 + 3 + 4 + 5))111$$

$$= (123)(124)(125)(134)(135)((1 + 2 + 3)/1)((1 + 2 + 3 + 4 + 5)/1)$$
Weighted enumeration of SST’s in shifted complexes

**Theorem** Let $\Lambda = \text{lk}_1 \Sigma$, $\tilde{\Lambda} = 1 \ast \Lambda$, $\Delta = \text{del}_1 \Sigma$, $\tilde{\Delta} = 1 \ast \Delta$.

$$h_d = \prod_{\sigma \in \Lambda} X_{\sigma \cup 1} \prod_{r} \left( \sum_{i=1}^{1+(d(\Delta)^T)_r} X_i / X_1 \right)$$

$$\quad = \prod_{\sigma \in \tilde{\Lambda}} X_{\sigma} \prod_{r} \left( \sum_{i=1}^{(d(\tilde{\Delta})^T)_r} X_i / X_1 \right).$$

**Example** bipyramid $\Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle$ again

$\Lambda = \text{lk}_1 \Sigma = \langle 23, 24, 25, 34, 35 \rangle$  $\tilde{\Lambda} = \langle 123, 124, 125, 134, 135 \rangle$

$\Delta = \text{del}_1 \Sigma = \langle 234, 235 \rangle$  $\tilde{\Delta} = \langle 1234, 1235 \rangle$

$$h_2 = (23)(24)(25)(34)(35)(1 + (2 + 3))(1 + (2 + 3 + 4 + 5))111$$

$$\quad = (123)(124)(125)(134)(135)((1 + 2 + 3)/1)((1 + 2 + 3 + 4 + 5)/1)$$
Fine weighting

- Weight $F = \{i_1 < \cdots < i_k\}$ by
  \[ x_{1,i_1} x_{2,i_2} \cdots x_{k,i_k}. \]

- Keeps track of where in each face the vertex appears.
- Can generalize our results on tree enumeration and eigenvalues, but things get more complex.
Conjecture for Matroid Complexes

\(h_d\) again seems to factor nicely, though we can’t describe it yet.
Cubical complexes

- To make boundary work in systematic way, take subcomplexes of high-enough dimensional cube (or, also possible to just define polyhedral boundary map).
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- Then we can define boundary map, and all the algebraic topology, including Laplacian.
- Analogues of Simplicial Matrix Tree Theorems follow readily (in fact for polyhedral complexes).
- Complete skeleta are very nicely behaved for eigenvalues, spanning trees.
- Cubical analogue of shifted complexes have integer eigenvalues; still working on trees.
Definition of color-shifted complexes

- Set of colors
- $n_c$ vertices, $(c, 1), (c, 2), \ldots (c, n_c)$ of color $c$.
- Faces contain at most one vertex of each color.
- Can replace $(c, j)$ by $(c, i)$ in a face if $i < j$.
Conjecture for complete color-shifted complexes

Let $\Delta$ be the color-shifted complex generated by the face with red $a$, blue $b$, green $c$. Let the red vertices be $x_1, \ldots, x_a$, the blue vertices be $y_1, \ldots, y_b$, and the green vertices be $z_1, \ldots, z_c$.

**Conjecture**

$$h_d(\Delta) = \left( \prod_{i=1}^{a} x_i \right)^{b+c-1} \left( \prod_{j=1}^{b} y_j \right)^{a+c-1} \left( \prod_{k=1}^{c} z_k \right)^{a+b-1}$$

$$\times \left( \sum_{i=1}^{a} x_i \right)^{(b-1)(c-1)} \left( \sum_{j=1}^{b} y_j \right)^{(a-1)(c-1)} \left( \sum_{k=1}^{c} z_k \right)^{(a-1)(b-1)}$$
Notes on conjecture

- This is with coarse weighting. Every vertex \( v \) has weight \( x_v \), and every face \( F \) has weight

\[
x_F = \prod_{v \in F} x_v.
\]

- The case with two colors is a (complete) Ferrers graph, studied by Ehrenborg and van Willigenburg.