A Simplicial matrix-tree theorem, II. Examples

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Definition of simplicial spanning trees

Let Δ be a d-dimensional simplicial complex.
$\Upsilon \subseteq \Delta$ is a **simplicial spanning tree** of Δ when:

0. $\Upsilon_{(d-1)} = \Delta_{(d-1)}$ ("spanning");
1. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
2. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Q}) = 0$ ("connected");
3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.

Metaconnectedness

- Denote by $\mathscr{T}(\Delta)$ the set of simplicial spanning trees of Δ.
- **Proposition** $\mathscr{T}(\Delta) \neq \emptyset$ iff Δ is **metaconnected** (homology type of wedge of spheres).
- Many interesting complexes are metaconnected, including everything we'll talk about.

Simplicial Matrix-Tree Theorem — Version II

- Δ^d = metaconnected simplicial complex
- $\Gamma \in \mathscr{T}(\Delta_{(d-1)})$
- ∂_Γ = restriction of ∂_d to faces not in Γ
- reduced Laplacian $L_\Gamma = \partial_\Gamma \partial_\Gamma^*$

Theorem [DKM, 2006]

$$h_d \;=\; \sum_{\Upsilon \in \mathscr{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon)|^2 \;=\; \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det L_\Gamma .$$

Weighted Simplicial Matrix-Tree Theorem — Version II

- Δ^d = metaconnected simplicial complex
- Introduce an indeterminate x_F for each face $F \in \Delta$
- Weighted boundary $\boldsymbol{\partial}$: multiply column F of (usual) ∂ by x_F
- $\Gamma \in \mathscr{T}(\Delta_{(d-1)})$
- $\boldsymbol{\partial}_\Gamma$ = restriction of $\boldsymbol{\partial}_d$ to faces not in Γ
- Weighted reduced Laplacian $\mathbf{L} = \boldsymbol{\partial}_\Gamma \boldsymbol{\partial}_\Gamma^*$

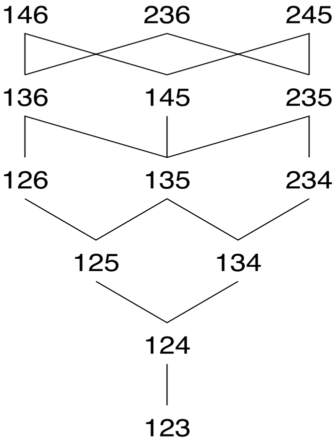
Theorem [DKM, 2006]

$$\mathbf{h}_d = \sum_{\Upsilon \in \mathscr{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon)|^2 \prod_{F \in \Upsilon} x_F^2 = \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det \mathbf{L}_\Gamma.$$

Definition of shifted complexes

- Vertices $1, \ldots, n$
- $F \in \Delta, i \notin F, j \in F, i < j \Rightarrow F \cup i - j \in \Delta$
- Equivalently, the k-faces form an initial ideal in the componentwise partial order.
- **Example** (bipyramid with equator)
  $\langle 123, 124, 125, 134, 135, 234, 235 \rangle$

Hasse diagram



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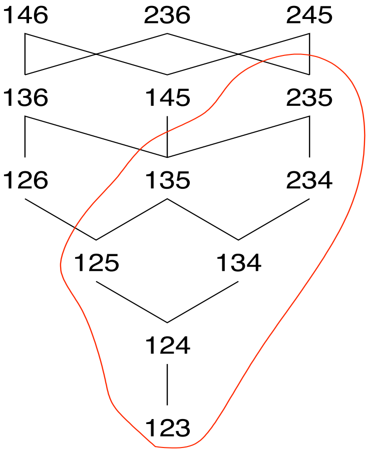

Links and deletions

- Deletion, $\operatorname{del}_1 \Delta = \{G \colon 1 \notin G, G \in \Delta\}$.
- Link, $\operatorname{lk}_1 \Delta = \{F - 1 \colon 1 \in F, F \in \Delta\}$.
- Deletion and link are each shifted, with vertices $2, \dots, n$.
- **Example**:

$$\begin{aligned} \Delta &= \langle 123, 124, 125, 134, 135, 234, 235 \rangle \\ \operatorname{del}_1 \Delta &= \langle 234, 235 \rangle \\ \operatorname{lk}_1 \Delta &= \langle 23, 24, 25, 34, 35 \rangle \end{aligned}$$

The Combinatorial fine weighting

Let Δ^d be a shifted complex on vertices $[n]$.
For each facet $A = \{a_1 < a_2 < \cdots < a_{d+1}\}$, define

$$x_A = \prod_{i=1}^{d+1} x_{i,a_i} \ .$$

Example If $\Upsilon = \langle 123, 124, 134, 135, 235 \rangle$ is a simplicial spanning tree of Δ, its contribution to $\mathbf{h}_2$ is

$$(x_{1,1}x_{2,2}x_{3,3})(x_{1,1}x_{2,2}x_{3,4})(x_{1,1}x_{2,3}x_{3,4})(x_{1,1}x_{2,3}x_{3,5})(x_{1,2}x_{2,3}x_{3,5})$$

From "Combinatorial" to "Algebraic"

- In Weighted Simplicial Matrix Theorem II, pick Γ to be the set of all $(d-1)$-dimensional faces containing vertex 1.

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$$\mathbf{h}_d = \det \mathbf{L}_\Gamma = \Big(\prod_{\sigma\in \mathrm{lk}_1 \Delta} \uparrow X_\sigma\Big) \det(X_{1,1} I + \hat{\mathbf{L}}_{\mathrm{del}_1 \Delta, d-1})$$

where $\hat{\mathbf{L}}$ is an "algebraic fined weighted Laplacian".

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$$= \Big(\prod_{\sigma\in \mathrm{lk}_1 \Delta} \uparrow X_\sigma\Big)\Big(\prod_{\substack{\lambda \text{ e'val of} \\ \hat{\mathbf{L}}_{\mathrm{del}_1 \Delta, d-1}}} X_{1,1} + \lambda\Big),$$

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The Algebraic fine weighted boundary map

For faces $A \subset B \in \Delta$ with $\dim A = i - 1$, $\dim B = i$, define

$$X_{AB} = \frac{\uparrow^{d-i} x_B}{\uparrow^{d-i+1} x_A}$$

where $\uparrow x_{i,j} = x_{i+1,j}$.

- Construct weighted boundary map $\boldsymbol{\partial}$ by multiplying (A, B) entry of usual boundary map ∂ by X_{AB}.
- Example:

$$X_{(235,25)} = \frac{x_{12}x_{23}x_{35}}{x_{22}x_{35}}$$

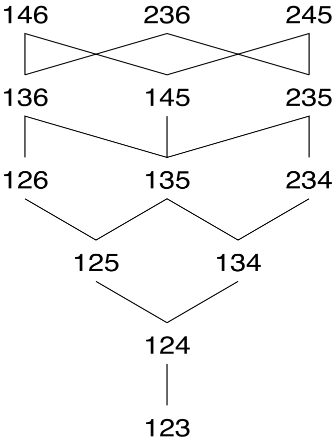
- Weighted boundary maps $\boldsymbol{\partial}$ satisfy $\boldsymbol{\partial\partial} = 0$.

Critical pairs

Definition A **critical pair** of a shifted complex Δ^d is an ordered pair (A, B) of $(d+1)$-sets of integers, where

- $A \in \Delta$ and $B \notin \Delta$; and
- B covers A in componentwise order.

Critical pairs



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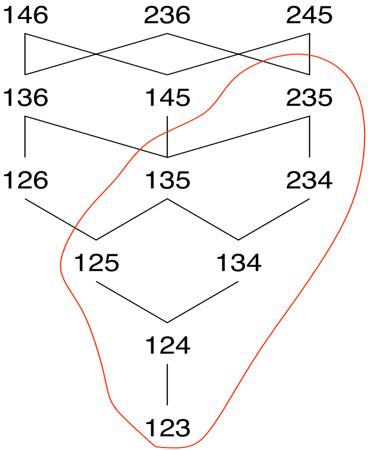

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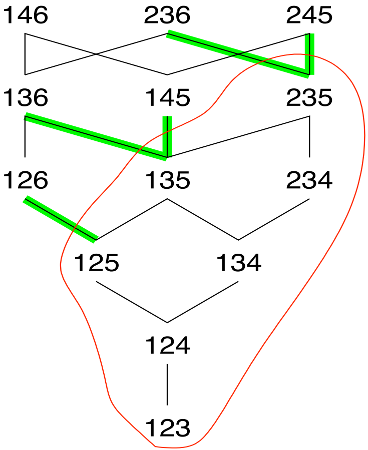

The Signature of a critical pair

Let (A, B) be a critical pair of a complex Δ:

$$A = \{a_1 < a_2 < \cdots < a_i < \cdots < a_{d+1}\},$$
$$B = A \setminus \{a_i\} \cup \{a_i + 1\}.$$

Definition The **signature** of (A, B) is the ordered pair

$$\big(\{a_1, a_2, \ldots, a_{i-1}\},\ a_i\big).$$

Example $\Delta = \langle 123, 124, 125, 134, 135, 234, 235\rangle$ (the bipyramid)

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(135,136)	
(135,145)	
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(235,245)	(2,3)

Finely Weighted Laplacian Eigenvalues

Theorem [DKM 2007]

Let Δ be a shifted complex.

Then the finely weighted Laplacian eigenvalues of Δ are specified completely by the signatures of critical pairs of Δ.

$$\text{signature } (S, a) \quad \leftrightarrow \quad \text{eigenvalue } \frac{1}{\uparrow X_S} \sum_{j=1}^{a} X_{S \cup j}$$

Examples of finely weighted eigenvalues

- Critical pair (135,145); signature (1,3):

$$\frac{X_{11}X_{21} + X_{11}X_{22} + X_{11}X_{23}}{X_{21}}$$

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$$\frac{X_{11}X_{22}X_{33} + X_{12}X_{22}X_{33} + X_{12}X_{23}X_{33} + X_{12}X_{23}X_{34} + X_{12}X_{23}X_{35}}{X_{22}X_{33}}$$

Corollaries

- Generalizes D.-Reiner formula for eigenvalues of shifted complexes in terms of degree sequences. (The "a" of the signatures are the entries of the conjugate degree sequence.)

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- Generalizes D.-Reiner formula for eigenvalues of shifted complexes in terms of degree sequences. (The "*a*" of the signatures are the entries of the conjugate degree sequence.)
- We can reconstruct a shifted complex from its finely weighted eigenvalues, so we can "hear the shape of a shifted complex", at least if our ears are fine enough.

Deletion-link recursion

We can compute signatures recursively, from deletion and link, as follows:

- Each (S, a) from $\operatorname{del}_1 \Delta$ is also a signature of Δ.
- Each (S, a) from $\operatorname{lk}_1 \Delta$ becomes signature $(S \cup 1, a)$ of Δ.
- Additionally, $\tilde{\beta}_{d-1}(\operatorname{del}_1 \Delta)$ copies of $(\emptyset, 1)$.

Example

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$\operatorname{del}_1 \Delta = \langle 234, 235 \rangle$	$\{(2,3), (23,5)\}$
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Finely weighted enumeration of SST's in shifted complexes

Theorem $\mathbf{h}_d = \left(\prod_{\sigma \in \mathrm{lk}_1 \Delta} X_{\sigma \cup 1}\right) \left(\prod_{(S,a) \in \mathrm{sign.}(\mathrm{del}_1 \Delta)} \frac{\sum_{j=1}^{a} X_{S \cup j}}{X_{S \cup 1}}\right).$

Example

$$\mathrm{lk}_1 \Delta = \langle 23, 24, 25, 34, 35 \rangle$$

$$\mathrm{del}_1 \Delta = \langle 234, 235 \rangle \qquad \mathrm{sign.}(\mathrm{del}_1 \Delta) = \{(2,3), (23,5)\}$$

$$\mathbf{h}_d(\Delta) = (X_{123} X_{124} X_{134} X_{125} X_{135}) \times \left(\frac{X_{12} + X_{22} + X_{23}}{X_{12}}\right) \left(\frac{X_{123} + X_{223} + X_{233} + X_{234} + X_{235}}{X_{123}}\right).$$

Corollary

By specializing to $d = 1$, we get a formula from Martin-Reiner (itself a special case of a result due to Remmel and Williamson) of finely weighted enumeration of spanning trees of threshold graphs (1-dimensional shifted complexes).

Definition of color-shifted complexes

- Set of colors
- n_c vertices, $(c,1),(c,2),\ldots(c,n_c)$ of color c.
- Faces contain at most one vertex of each color.
- Can replace (c,j) by (c,i) in a face if $i<j$.
- Example: Faces written as (red,blue,green): 111, 112, 113, 121, 122, 123, 131, 132, 211, 212, 213, 221, 222, 223, 231,232.

Conjecture for complete color-shifted complexes

Let Δ be the color-shifted complex generated by the face with red a, blue b, green c. Let the red vertices be $x_1, \ldots, x_a$, the blue vertices be $y_1, \ldots, y_b$, and the green vertices be $z_1, \ldots, z_c$.

Conjecture

$$\mathbf{h}_d(\Delta) = (\prod_{i=1}^{a} x_i)^{b+c-1}(\prod_{j=1}^{b} y_j)^{a+c-1}(\prod_{k=1}^{c} z_k)^{a+b-1}$$
$$\times (\sum_{i=1}^{a} x_i)^{(b-1)(c-1)}(\sum_{j=1}^{b} y_j)^{(a-1)(c-1)}(\sum_{k=1}^{c} z_k)^{(a-1)(b-1)}$$

Notes on conjecture

- This is with coarse weighting. Every vertex v has weight x_v, and every face F has weight

$$x_F = \prod_{v \in F} x_v.$$

- The case with two colors is a (complete) **Ferrers graph**, studied by Ehrenborg and van Willigenburg.

Conjecture for Matroid Complexes

- $\mathbf{h}_d$ again seems to factor nicely, though we can't describe it yet.
- Once again, with coarse weighting