The Importance of being Equivalent:

The Ubiquity of equivalence relations in mathematics, K-16+

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Equality

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Along with operations (such as $+$),

$$\begin{array}{rc} \text{if} & a = b \\ \text{and} & c = d \\ \hline \text{then} & a + c = b + d \end{array}$$

(substitution)

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- If $\frac{2}{3}$ and $\frac{10}{15}$ are equal, why can we use one but not the other?
- Could we have used something else besides $\frac{10}{15}$?
- Would we use something else in another situation, or should we always use $\frac{10}{15}$?

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Because of these three properties, we say $\sim$ is an equivalence relation

Partitioning fractions

Because fraction equivalence is an equivalence relation, we can partition fractions as follows:

$\frac{a}{b}$ and $\frac{c}{d}$ are in the same part ("equivalence class") if $\frac{a}{b} \sim \frac{c}{d}$.

$\frac{1}{2}$	$\frac{17}{34}$	$\frac{2}{3}$	$\frac{10}{15}$	$\frac{1}{5}$	$\frac{3}{15}$	$\frac{4}{7}$	$\frac{20}{35}$
$\frac{4}{8}$	$\frac{6}{12}$	$\frac{4}{6}$	$\frac{14}{21}$	$\frac{10}{50}$	$\frac{8}{40}$	$\frac{40}{70}$	$\frac{16}{28}$
$\frac{10}{20}$	$\frac{7}{14}$	$\frac{20}{20}$	$\frac{8}{12}$	$\frac{2}{10}$	$\frac{7}{35}$	$\frac{8}{14}$	$\frac{36}{63}$

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Rules for partitions:

- Everything is in exactly one part
- No empty part

Adding fractions (revisited)

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- In other settings, we stick to the fraction in lowest terms, a distinguished representative.

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- Same shape, size, chirality, orientation
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- Same shape, size, chirality, orientation, position
  - equality

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- Different situations call for different interpretations of when two shapes are "the same".

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Names

Gwendolen: ... my ideal has always been to love some one of the name of Ernest. There is something in that name that inspires absolute confidence. The moment Algernon first mentioned to me that he had a friend called Ernest, I knew I was destined to love you.

Cecily: ... it had always been a girlish dream of mine to love some one whose name was Ernest. There is something in that name that seems to inspire absolute confidence. I pity any poor married woman whose husband is not called Ernest.

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- At parking meters, quarters good, everything else bad.
- Everywhere, pennies bad.

Fractions, again

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- When it's apple pie, and you have two kids

Fractions, again

When is $\frac{2}{6}$ not the same as $\frac{1}{3}$?

- When it's apple pie.
- When it's apple pie, and you have two kids and no knife.

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- refine partitions

Where else do we see this?

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Glad you asked

Regrouping

To do multidigit addition and subtraction,

$$436 = 400 + 30 + 6 = 400 + 20 + 16 = 300 + 130 + 6 = \cdots$$

- Different representations are better or worse for different addition and subtraction problems.
- Using base-10 blocks, these all make different (but "equivalent") pictures.

"Unique" factorization

Completely factor 60, as

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- Distinguished representative is usually to arrange primes from smallest to largest.
- In context of factorization, 6×10 and 4×15 are different, even though usually $6 \times 10 = 4 \times 15$.

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- So it's close enough for everything we do.
- And allowing it (and all its infinite process buddies) allows us to say things like $\sqrt{2}$ and e are numbers, on the number line.

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- But it is not so obvious when expressions are equivalent.
- There are many different ideas of "distinguished representative".

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- Having the same solution set for [all relevant variables] is an equivalence relation.
- The algebraic manipulations we do when solving equations should take us from equations only to equivalent equations.

Elementary Probability (combinations and permutations)

When you ask "How many ways can we pick 6 of these 54 numbers?" [Texas Lotto], we mean $\{17, 23, 42, 10, 54, 1\}$ is the same as $\{10, 23, 54, 17, 42, 1\}$,

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- Thinking of combinations as an equivalence relation on permutations allows us to get counting formula for combinations.
- To present a combination, we need to pick some way of writing it down (a permutation), a representative of its equivalence class.
- Usually, the distinguished representative (ordered list) to represent a combination (unordered list) is to put the items "in order"; for instance: $\{1, 10, 17, 23, 42, 54\}$.

More about counting

The relation between counting formulas for permutations and combinations reminds us of one more thing equivalence relations are good for (that doesn't show up in the elementary examples): If the equivalence classes all have the same number of elements (perhaps by some symmetry argument), then

$$\text{size of set} = (\text{number of classes}) \times (\text{size of classes})$$

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- This equivalence relation respects vector addition and scalar multiplication.

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- Example: Last digit arithmetic ($m = 10$).

Anti-differentiation

Solve

$$f'(x) = 3x^2$$

- "Answer" is $x^3 + C$.
- This really means the equivalence class of functions that can be written in this form.
- The equivalence relation is $f \sim g$ if $f - g$ is a constant.
- This equivalence relation respects addition, multiplication by a constant, which is why those are easy to deal with in anti-differentiation.

Linear Differential equations

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$$y''' - 5y'' + y' - y = 3x^2$$

- Solutions of the form

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- This really means the equivalence class of functions that can be written in this form.
- The equivalence relation is $f \sim g$ if $f - g$ is a solution of the homogeneous equation.

Similarly for the matrix equation

$$Mx = b.$$

Gaussian elimination in matrices

- Consists of a series of elementary row operations that do not change the solution set.
- So at the end, we have a nicer representative of the same equivalence class (of systems with the same solution).

Cardinality

What is the cardinality of a set?

- It's not defined as a function, *per se*
- We just say when two sets have the same cardinality.
- That's an equivalence relation, not a function.
- There are some distinguished representatives: $0; 1; 2; \ldots; \mathbb{N}; \mathbb{R}$.

Isomorphisms

- Graphs, groups, topological spaces, partial orders, etc.
- Two objects are isomorphic if they have the same structure that we care about, even though they may look very different.
- It can be difficult to determine when two objects are isomorphic.

Why do some equivalence relations respect addition?

What we really need is to make sure that $[0]$ acts like the additive identity:

$$[0] + [0] = [0].$$

Also

$$-[0] = [0].$$

This is just the definition of subgroup (in an abelian group).

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Similarly, the nonabelian case gives rise to normal subgroups.

Why do some equivalence relations respect multiplication?

What we really need is to make sure that [0] acts like the multiplicative “killer”:

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for all $[x]$.
Along with the subgroup condition (for addition), this is just the definition of ideal.

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If these hold, then it’s easy to check that the equivalence relation respects multiplication.

Why do some equivalence relations respect order?

- What we really need is to make sure positive $\times$ positive is positive, positive $+$ positive is positive.
- If these hold, it's easy to check our usual rules about addition and multiplication respecting order.

Well-defined functions

- Functions on equivalence classes are often defined in terms of first picking a representative.
- Operation-preserving is a special case of this.
- We have to make sure it doesn't matter which representative we pick.
- In the algebraic setting, this usually reduces to checking that $f([0]) = 0$.