The Importance of being Equivalent:
The Ubiquity of equivalence relations in mathematics, K-16+

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Math Education Seminar
University of Kentucky
March 31, 2011
Equality

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- if $a = b$ and $b = c$, then $a = c$ (transitivity)
  - (so we can write things like $a = b = c$)
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Along with operations (such as +),

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\text{if } a = b \\
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Along with operations (such as $+$),

\[
\text{if } a = b \quad \text{and} \quad c = d \quad \text{then} \quad a + c = b + d
\]

(substitution)
One reason fractions are hard

\[
\frac{2}{3} + \frac{1}{5} =
\]
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\frac{2}{3} + \frac{1}{5} = \frac{10}{15} + \frac{3}{15}
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We have to use \(\frac{2}{3} = \frac{10}{15}\) and \(\frac{1}{5} = \frac{3}{15}\).
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Questions:

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▶ If \(\frac{2}{3}\) and \(\frac{10}{15}\) are equal, why can we use one but not the other?
▶ Could we have used something else besides \(\frac{10}{15}\)?
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Questions:

- If \(\frac{2}{3}\) and \(\frac{10}{15}\) are equal, why can we use one but not the other?
- Could we have used something else besides \(\frac{10}{15}\)?
- Would we use something else in another situation, or should we always use \(\frac{10}{15}\)?
Equivalent fractions

Definition: \( \frac{a}{b} \sim \frac{c}{d} \) if they reduce to the same fraction \((ad = bc)\). It’s easy to check the following properties:
Equivalent fractions

Definition: $\frac{a}{b} \sim \frac{c}{d}$ if they reduce to the same fraction ($ad = bc$). It’s easy to check the following properties:

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Because of these three properties, we say \( \sim \) is an equivalence relation.
Partitioning fractions

Because fraction equivalence is an equivalence relation, we can partition fractions as follows:

\[ \frac{a}{b} \text{ and } \frac{c}{d} \text{ are in the same part ("equivalence class") if } \frac{a}{b} \sim \frac{c}{d}. \]

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| \( \frac{4}{8} \times \frac{6}{12} \) | \( \frac{4}{6} \times \frac{14}{21} \) | \( \frac{10}{50} \times \frac{8}{40} \) | \( \frac{40}{70} \times \frac{16}{28} \) |
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Rules for partitions:

- Everything is in exactly one part
- No empty part
Adding fractions (revisited)

\[
\text{if } \frac{a}{b} \sim \frac{c}{d} \quad \text{and} \quad \frac{e}{f} \sim \frac{g}{h}
\]

\[
\text{then } \frac{a}{b} + \frac{e}{f} \sim \frac{c}{d} + \frac{g}{h}
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So, really we should say \( \frac{2}{3} + \frac{1}{5} = \frac{13}{15} \), because anything equivalent to \( \frac{2}{3} \) plus anything equivalent to \( \frac{1}{5} \) "equals" something equivalent to \( \frac{13}{15} \).

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\frac{a}{b} + \frac{e}{f} \sim \frac{c}{d} + \frac{g}{h}
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So, really we should say

\[
\begin{bmatrix}
2 \\
3
\end{bmatrix} + \begin{bmatrix}
1 \\
5
\end{bmatrix} = \begin{bmatrix}
13 \\
15
\end{bmatrix},
\]

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Similarity, congruence, etc.

Some equivalence relations from geometry:
Similiarity, congruence, etc.

Some equivalence relations from geometry:

- **Similarity**
  - same “shape”, possibly different size
  - can get via dilation, reflection, rotation, translation

- **Congruence**
  - same “shape”, size
  - can get via reflection, rotation, translation
  - same shape, size, chirality
  - can get via rotation, translation
  - same shape, size, chirality, orientation
  - can get via translation
  - same shape, size, chirality, orientation, position
  - equality
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Finer partitions

As we go down that ladder, we refine the partition, by splitting each part into more parts.

Different situations call for different interpretations of when two shapes are “the same”.

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Finer partitions

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Finer partitions

- As we go down that ladder, we refine the partition, by splitting each part into more parts.
- Different situations call for different interpretations of when two shapes are “the same”.
Cars

- My car is different than yours (not equal), even if they are the same model.
Cars

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- But if they are the same model, they have the same maintenance schedule.
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- But even then, they may not be repaired on the same schedule.
Cars

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▶ But if they are the same model, they have the same maintenance schedule.
▶ But even then, they may not be repaired on the same schedule.
▶ “A Japanese car needs you to hold the handle when you lock it, but an American car does not.”
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  - Which Japanese car?
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- But even then, they may not be repaired on the same schedule.
- “A Japanese car needs you to hold the handle when you lock it, but an American car does not.”
  - Which Japanese car?
  - Which American car?
Names

Gwendolen: ... my ideal has always been to love some one of the name of Ernest. There is something in that name that inspires absolute confidence. The moment Algernon first mentioned to me that he had a friend called Ernest, I knew I was destined to love you.

Cecily: ... it had always been a girlish dream of mine to love some one whose name was Ernest. There is something in that name that seems to inspire absolute confidence. I pity any poor married woman whose husband is not called Ernest.
Money

- At the store, 1 dollar equals 4 quarters equals 10 dimes.
Money

- At the store, 1 dollar equals 4 quarters equals 10 dimes.
- At old vending machines, dollar bad, coins good.
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- At my vending machine, dollar good, coins bad.
- At parking meters, quarters good, everything else bad.
Money

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- At old vending machines, dollar bad, coins good.
- At my vending machine, dollar good, coins bad.
- At parking meters, quarters good, everything else bad.
- Everywhere, pennies bad.
Fractions, again

When is $\frac{2}{6}$ not the same as $\frac{1}{3}$?
Fractions, again

When is $\frac{2}{6}$ not the same as $\frac{1}{3}$?

- When it’s apple pie.
Fractions, again

When is \( \frac{2}{6} \) not the same as \( \frac{1}{3} \)?

- When it’s apple pie.
- When it’s apple pie, and you have two kids.
Fractions, again

When is \( \frac{2}{6} \) not the same as \( \frac{1}{3} \)?

- When it’s apple pie.
- When it’s apple pie, and you have two kids and no knife.
What have we seen equivalences do?

▶ recognize and categorize
What have we seen equivalences do?

- recognize and categorize
  - by partition (maybe by distinguished representative)
What have we seen equivalences do?

- recognize and categorize
  - by partition (maybe by distinguished representative)
  - by transitivity and basic equivalences
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- define functions (have to be well-defined)
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- recognize and categorize
  - by partition (maybe by distinguished representative)
  - by transitivity and basic equivalences
- respect operations (make operations easier)
- define functions (have to be well-defined)
- refine partitions
Where else do we see this?
Where else do we see this?

Glad you asked
Regrouping

To do multidigit addition and subtraction,

\[ 436 = 400 + 30 + 6 = 400 + 20 + 16 = 300 + 130 + 6 = \cdots \]

- Different representations are better or worse for different addition and subtraction problems.
- Using base-10 blocks, these all make different (but "equivalent") pictures.
“Unique” factorization

Completely factor 60, as

\[ 2 \times 2 \times 3 \times 5 = 2 \times 3 \times 2 \times 5 = 5 \times 2 \times 2 \times 3 = \cdots \]

Natural to say these are all the “same”; once we do, we get unique factorization into primes.
“Unique” factorization

Completely factor 60, as

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▶ Natural to say these are all the “same”; once we do, we get unique factorization into primes.

▶ Distinguished representative is usually to arrange primes from smallest to largest.

▶ In context of factorization, \(6 \times 10\) and \(4 \times 15\) are different, even though usually \(6 \times 10 = 4 \times 15\).
0.999... = 1
0.999... 

right?

0.999... = 1
0.999\ldots

0.999\ldots = 1

right?

- 0.999\ldots isn’t even a number, it’s an infinite process
0.999... right?

- 0.999... isn’t even a number, it’s an infinite process that gets arbitrarily close to 1.
0.999\ldots

0.999\ldots = 1

right?

- 0.999\ldots isn’t even a number, it’s an infinite process that \textit{gets arbitrarily close to} 1
- “gets arbitrarily close to” is an equivalence relation.
0.999... 

right?

▶ 0.999... isn’t even a number, it’s an infinite process that gets arbitrarily close to 1

▶ “gets arbitrarily close to” is an equivalence relation.

▶ This equivalence relation respects addition, multiplication, etc. (like equivalent fractions).
0.999... = 1

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▶ 0.999... isn’t even a number, it’s an infinite process that gets arbitrarily close to 1
▶ “gets arbitrarily close to” is an equivalence relation.
▶ This equivalence relation respects addition, multiplication, etc. (like equivalent fractions).
▶ So it’s close enough for everything we do.
0.999... is even a number, it’s an infinite process that gets arbitrarily close to 1.

“gets arbitrarily close to” is an equivalence relation.

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So it’s close enough for everything we do.

And allowing it (and all its infinite process buddies) allows us to say things like \( \sqrt{2} \) and \( e \) are numbers, on the number line.
Algebraic expressions

$$(x - 1)(x + 1) = x^2 - 1$$
Algebraic expressions

\[(x - 1)(x + 1) = x^2 - 1\]

right?
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▶ The two expressions are equal for all values of \(x\).
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\[(x - 1)(x + 1) = x^2 - 1\]

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▶ The two expressions are equal for all values of \(x\).
▶ Being equal for all values of \([\text{all relevant variables}]\) is an equivalence relation.
Algebraic expressions

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- So it's good enough for everything we do.
- But it is not so obvious when expressions are equivalent.
Algebraic expressions

\[(x - 1)(x + 1) = x^2 - 1\]

right?

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- Being equal for all values of [all relevant variables] is an equivalence relation.
- This equivalence relation respects addition, multiplication, etc. (like equivalent fractions).
- So it’s good enough for everything we do.
- But it is not so obvious when expressions are equivalent.
- There are many different ideas of “distinguished representative”.

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The Importance of being Equivalent:
Algebraic equations

$3x + 7 = 22$ is the same as $3x = 15$, right?

The two equations have the same solution set for $x$.

Having the same solution set for all relevant variables is an equivalence relation.

The algebraic manipulations we do when solving equations should take us from equations only to equivalent equations.
Algebraic equations

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- Having the same solution set for [all relevant variables] is an equivalence relation.
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- Having the same solution set for [all relevant variables] is an equivalence relation.
- The algebraic manipulations we do when solving equations should take us from equations only to equivalent equations.
Elementary Probability (combinations and permutations)

When you ask “How many ways can we pick 6 of these 54 numbers?” [Texas Lotto], we mean \{17, 23, 42, 10, 54, 1\} is the same as \{10, 23, 54, 17, 42, 1\},
Elementary Probability (combinations and permutations)

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- Thinking of combinations as an equivalence relation on permutations allows us to get counting formula for combinations.
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- To present a combination, we need to pick some way of writing it down (a permutation), a representative of its equivalence class.
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- Thinking of combinations as an equivalence relation on permutations allows us to get counting formula for combinations.
- To present a combination, we need to pick some way of writing it down (a permutation), a representative of its equivalence class.
- Usually, the distinguished representative (ordered list) to represent a combination (unordered list) is to put the items “in order”; for instance: \{1, 10, 17, 23, 42, 54\}.

Art Duval

The Importance of being Equivalent:
More about counting

The relation between counting formulas for permutations and combinations reminds us of one more thing equivalence relations are good for (that doesn’t show up in the elementary examples): If the equivalence classes all have the same number of elements (perhaps by some symmetry argument), then

$$\text{size of set} = (\text{number of classes}) \times (\text{size of classes})$$
Vectors

- To draw a vector in the plane, we need to pick a starting point and ending point for the arrow.
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- So we can think of a vector as an equivalence class of arrows; two arrows are equivalent if they have the same direction and magnitude.
- Distinguished representative is often to start at the origin. But to see how to add two vectors, we should move the starting point of the second one.
- This equivalence relation respects vector addition and scalar multiplication.
Modular arithmetic

- Two numbers are equivalent if they give the same remainder after dividing by \( m \).
- Example: Even and odd \((m = 2)\).
Modular arithmetic

- Two numbers are equivalent if they give the same remainder after dividing by $m$.
- Example: Even and odd ($m = 2$).
- This equivalence relation respects addition and multiplication.
- Example: Last digit arithmetic ($m = 10$).
Anti-differentiation

Solve

\[ f'(x) = 3x^2 \]

- “Answer” is \( x^3 + C \).
- This really means the equivalence class of functions that can be written in this form.
- The equivalence relation is \( f \sim g \) if \( f - g \) is a constant.
- This equivalence relation respects addition, multiplication by a constant, which is why those are easy to deal with in anti-differentiation.
Linear Differential equations

Solve

\[ y''' - 5y'' + y' - y = 3x^2 \]

- Solutions of the form

\[ y = y_0 + y_p \]

where \( y_0 \) is the general solution to the homogeneous equation, and \( y_p \) is a particular solution.

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Similarly for the matrix equation

\[ Mx = b. \]
Gaussian elimination in matrices

- Consists of a series of elementary row operations that do not change the solution set.
- So at the end, we have a nicer representative of the same equivalence class (of systems with the same solution).
Cardinality

What is the cardinality of a set?

- It’s not defined as a function, \textit{per se}
- We just say when two sets have the same cardinality.
- That’s an equivalence relation, not a function.
- There are some distinguished representatives: $0; 1; 2; \ldots; \mathbb{N}; \mathbb{R}$. 
Isomorphisms

- Graphs, groups, topological spaces, partial orders, etc.
- Two objects are isomorphic if they have the same structure that we care about, even though they may look very different.
- It can be difficult to determine when two objects are isomorphic.
Why do some equivalence relations respect addition?

What we really need is to make sure that \([0]\) acts like the additive identity:

\[ [0] + [0] = [0]. \]

Also

\[ -[0] = [0]. \]

This is just the definition of subgroup (in an abelian group).
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Similarly, the nonabelian case gives rise to normal subgroups.
Why do some equivalence relations respect multiplication?

What we really need is to make sure that \([0]\) acts like the multiplicative “killer”:

\[ [0] \times [x] = [0] \]

for all \([x]\).
Along with the subgroup condition (for addition), this is just the definition of ideal.
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If these hold, then it’s easy to check that the equivalence relation respects multiplication.
Why do some equivalence relations respect order?

- What we really need is to make sure positive $\times$ positive is positive, positive $+$ positive is positive.
- If these hold, it’s easy to check our usual rules about addition and multiplication respecting order.
Well-defined functions

- Functions on equivalence classes are often defined in terms of first picking a representative.
- Operation-preserving is a special case of this.
- We have to make sure it doesn’t matter which representative we pick.
- In the algebraic setting, this usually reduces to checking that $f([0]) = 0$. 