Max flow min cut in higher dimensions

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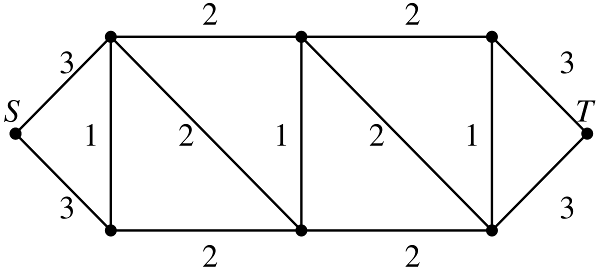
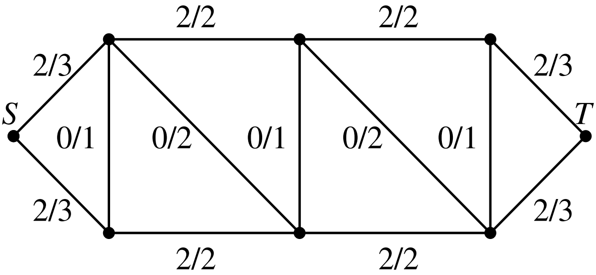
Max flow



Given a graph G with source S, sink T, and edge capacities κ.

Max flow



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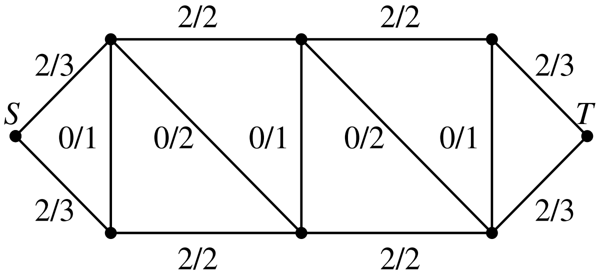
Definition

Flow on G is an assignment of flow x_e (non-negative number, and direction) to each edge such that:

- net flow at each vertex, except S and T, is zero; and
- $|x_e| \leq \kappa_e$.

Value of flow is outflow(S) = inflow(T).

Max flow



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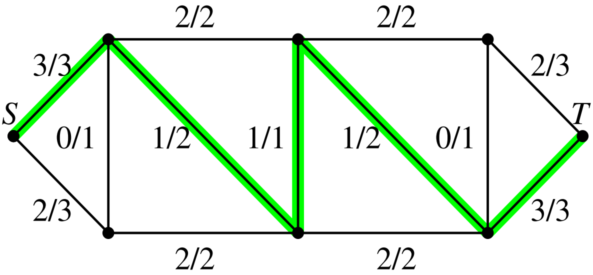

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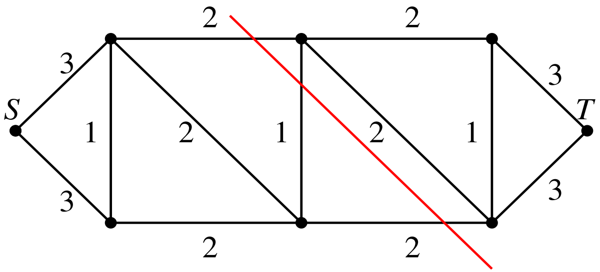

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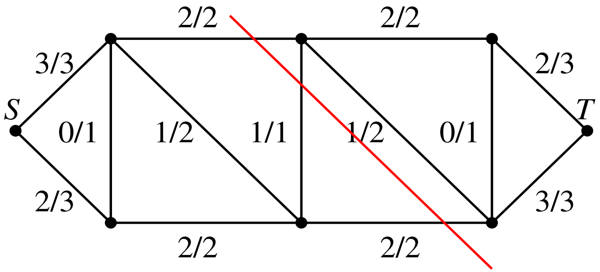
Cut is minimal set of edges whose removal disconnects S from T.
Value of cut is $\sum_{e \in \text{cut}} \kappa_e$.

Min cut



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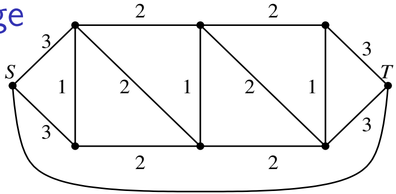
Clearly, value(flow) $\leq$ value(cut), so max flow $\leq$ min cut.

Theorem (Classic max flow min cut)

Max flow = min cut.

Add an extra edge



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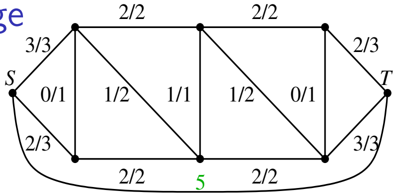

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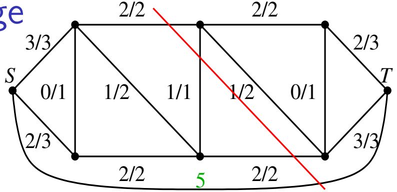

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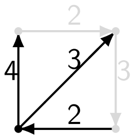
Cut is minimal set of edges, including e_0, whose removal disconnects G. Value of cut is $\sum_{e \in \mathrm{cut} \setminus e_0} \kappa_e$.

Flows and boundary



$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \\ 2 \end{pmatrix} = -4 - 3 + 2 = -5$$

Assign orientation to each edge (flow going "backwards" gets negative value)

$$\mathrm{netflow}(v) = \sum_{v=e^+} x_e - \sum_{v=e^-} x_e = \sum_{v \in e} (-1)^{\varepsilon(e,v)} x_e = (\partial x)_v$$

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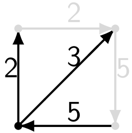

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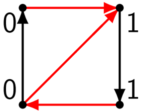
So net flow condition is $\partial x = 0$.

Cuts and coboundary



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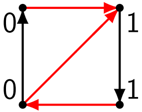

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$$\partial^T y = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}^T \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & -1 \end{pmatrix}$$

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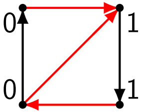

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Edges in cut are those that have both 0 and 1 endpoints.

Linear programming

Flow is now a linear program

- Find vector x (in edge space)
- $\partial x = 0$ (x is in flow space)
- $-\kappa_e \leq x_e \leq \kappa_e$ (can omit e_0)
- max x_0

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The dual program is (can easily be reworked to say):

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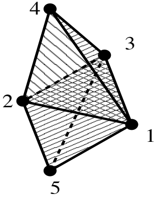
Linear programming says the solutions are equal; with some effort we can show the solution to the dual LP is the min cut problem.

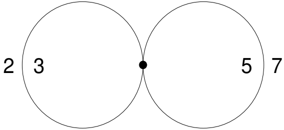
Cell complexes

Definition

A cell complex X is a finite CW-complex (i.e., collection of cells of different dimensions), with say n facets and p ridges, and a $p \times n$ cellular boundary matrix $\partial \in \mathbb{Z}^{p \times n}$.

Example





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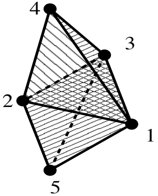

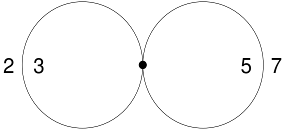

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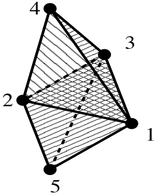
Remark

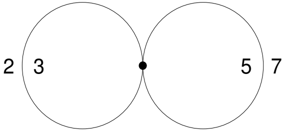
Any $\mathbb{Z}$ matrix can be the boundary matrix of a cell complex.

Cellular matroids

- Matroid whose elements are columns of boundary matrix
- Dependent sets are the supports of the kernel of the boundary matrix

Example





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Flow space and circuits

Definition

i-dimensional flow space of cell complex X is

$$\mathrm{Flow}_i(X) = \ker(\partial_i : C_{i-1}(X, \mathbb{R}) \to C_{i-1}(X, \mathbb{R})).$$

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A circuit of X is a minimal set of i-faces that support non-0 vector of $Flow_i(X)$

Remark

Circuits are the circuits (minimal dependent sets) of cellular matroid.

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Example

Bipyramid

Cut space and cocircuits

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i-dimensional cut space of cell complex X is

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Cut space is the rowspace of the boundary matrix; cut space and flow space are orthogonal complements.

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Topological interpretation of cocircuits

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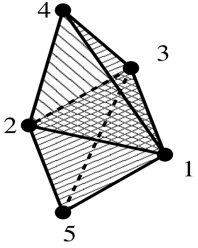
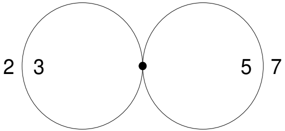
Cocircuits are minimal for increasing $(i-1)$-dimensional homology instead of decreasing i-dimensional homology

Examples





Max flow in higher dimensions

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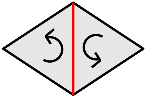
Find a codimension-1 cycle on the complex, and attach a facet f_0 filling that cycle. We are trying to maximize circulation on that designated facet (around that cycle), while making all circulation balance on each codimension-1 face (ridge).



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Theorem (DKM)

The max flow equals the value of some solution to the dual LP whose support is a cocircuit.

Summary

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Fine print:

- normalize cut vector by specifying its value is 1 on f_0, the added filling-in facet
- cut vector might not be all 1's and 0's
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Questions:

- Is there an analogue to Ford-Fulkerson? That is, a combinatorial algorithm that would construct the "min cut", without relying on linear programming?
- What happens when we restrict to integers?