Cuts and flows in cell complexes, I: Topology and vector space bases

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Cuts and bonds

Let $G$ be a connected graph.

**Definition**
A **cut** is a collection of edges in $G$ whose removal disconnects the graph;

**Example**

![Graph example](image_url)
Cuts and bonds

Let $G$ be a connected graph.

**Definition**
A **cut** is a collection of edges in $G$ whose removal disconnects the graph; a **bond** is a minimal cut.

**Example**

![Diagram of a connected graph with two cuts and a bond]

**Remark**
Using matroid language, bonds are cocircuits.
Orientation

Removal of a bond leaves two connected “shores” of the graph. This gives an alternative way to define a bond: describe the partition of the vertices of the graph. We can orient all edges of the bond so they point from, say, north to south.

Example

\[ \text{Diagram of a square graph with a bond removed, showing two connected shores.} \]
Orientation

Removal of a bond leaves two connected "shores" of the graph. This gives an alternative way to define a bond: describe the partition of the vertices of the graph. We can orient all edges of the bond so they point from, say, north to south.

**Example**

![Graph](image)

In fact, we can get this orientation from the partition: Take the coboundary (directed edges pointing out) of all vertices in, say, north shore. All edges completely within shore cancel out, leaving only those edges coming out of the north shore.
Cut space

This suggests looking at cuts and bonds as the image of the coboundary. This is a vector space over field (\(\mathbb{R}\) or \(\mathbb{Q}\)).
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Definition

Cut space of \(G\) is image of coboundary, \(\text{im} \partial^*\), i.e., row-span of boundary [incidence] matrix.
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**Definition**

Cut space of $G$ is image of coboundary, $\text{im} \partial^*$, i.e., row-span of boundary [incidence] matrix.

**Example**

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 \\
-1 & 1 & 0 & 0 & 1 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & -1 & -1
\end{pmatrix}
\]

Sum of first two rows ($\partial^*$ of north shore) is supported on bond.
Cut space

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Sum of first two rows ($\partial^*$ of north shore) is supported on bond.

**Question**

What is a basis?
Fundamental bond

Definition
Given a spanning tree \( T \)

Example
![Diagram showing a spanning tree with a fundamental bond highlighted]
Fundamental bond

**Definition**
Given a spanning tree $T$ and an edge $e \in T$, the fundamental bond is the unique bond containing $e$, and no other edge from $T$.

**Example**

![Example diagram](image)
Fundamental bond

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Example

Theorem
For a fixed spanning tree, the collection of fundamental bonds forms a basis of cut space.
Flows and circuits

Definition

A circuit is a closed path with no repeated vertices.
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In matroid terms, a circuit is a minimal dependent set, and dependent sets are in kernel of boundary, so it is natural to define...
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Flow space of $G$ is kernel of boundary matrix
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**Definition**

Given a spanning tree $T$

**Example**

![Diagram of a graph with a spanning tree and a fundamental circuit highlighted.]
Fundamental circuit

**Definition**
Given a spanning tree $T$ and an edge $e \not\in T$, the fundamental circuit is the unique circuit in $T \cup \{e\}$.

**Example**

![Diagram](image-url)
Fundamental circuit

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**Example**

![Diagram of fundamental circuits](image)

**Theorem**
For a fixed spanning tree, the collection of fundamental circuits forms a basis of flow space.
Cell complexes

Definition
A cell complex $X$ is a finite CW-complex (i.e., collection of cells of different dimensions),
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Think the boundary of each facet being a $\mathbb{Z}$-linear combination of ridges.

Remark
Any $\mathbb{Z}$ matrix can be the boundary matrix of a cell complex.
Examples

\begin{align*}
\begin{array}{cccc}
2 & 3 & 0 & 0 \\
0 & 0 & 5 & 7 \\
\end{array}
\end{align*}

\begin{align*}
\begin{array}{ccc}
0 & -2 & 2 \\
1 & 0 & -2 \\
-1 & 2 & 0 \\
\end{array}
\end{align*}
Cellular matroids

- Matroid whose elements are columns of boundary matrix

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- Dependent sets are the supports of the kernel of the boundary matrix
Cellular matroids

- Matroid whose elements are columns of boundary matrix
- Dependent sets are the supports of the kernel of the boundary matrix
- Spanning trees are bases of these matroids (maximal independent sets)
Spanning forests (Bolker; Kalai; DKM)

A Cellular spanning forest (CSF) is $\Upsilon \subset X$ such that:
$\Upsilon_{(d-1)} = X_{(d-1)}$ (same $(d-1)$-skeleton),
Spanning forests (Bolker; Kalai; DKM)

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- Equivalently, $\{\partial F : F \in \mathcal{Y}\}$ is a vector space basis for $\text{im} \partial$
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A Cellular spanning forest (CSF) is $\gamma \subset X$ such that:
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- Equivalently, $\{\partial F : F \in \gamma\}$ is a vector space basis for $\text{im} \partial$
Cut space and bonds

Definition

$i$-dimensional cut space of cell complex $X$ is

$$\text{Cut}_i(X) = \text{im}(\partial_i^*: C_{i-1}(X, \mathbb{R}) \rightarrow C_i(X, \mathbb{R})).$$

Remark

Cut space is the rowspace of the boundary matrix.
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A bond of $X$ is a minimal set of $i$-faces that support non-0 vector of $\text{Cut}_i(X)$

Remark

Cut space is the rowspace of the boundary matrix.

Remark

Bonds are the cocircuits of cellular matroid
Topological interpretation of bonds

Remark
Bonds are minimal for increasing \((i - 1)\)-dimensional homology instead of decreasing \(i\)-dimensional homology

Examples
Characteristic vectors of bonds

Fix bond $B$

**Proposition**

$\text{Cut}_B(X) := (\{0\} \cup (\text{Cut}_i(X) \cap \{v : \text{supp}(v) = B\}))$ is $1$-dimensional

**Example**
Topological interpretation of characteristic vector

Example

If $B = \{F_5, F_7\}$, then $\text{Cut}_B$ spanned by $5F_5 + 7F_7$. 

Topological interpretation of characteristic vector

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Theorem (DKM)

Let $A$ be a cellular spanning forest of $X/B$. Then $\text{Cut}_B(X)$ is spanned by

$$\chi(B, A) := \sum_{F \in B} \pm|\tilde{H}(A \cup F, \mathbb{Z})|$$
Topological interpretation of characteristic vector

Example

If $B = \{F_5, F_7\}$, then $\chi(B, F_2) = 2(5F_5 + F_7)$, but $\chi(B, F_3) = 3(5F_5 + F_7)$.

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Definition

The characteristic vector of $B$ is $\chi(B, A)$
Orientation

One way to get at the sign in $\chi(B, A)$: Oriented matroid theory says that for every pair of elements $F, F'$ of a cocircuit $B$, there is a unique circuit $C$ such that $C \cap B = \{F, F'\}$, and we can deduce the relative signs on $F, F'$ in $\chi$ by their relative signs in $C$. 
**Fundamental bond**

**Definition**
Given a spanning forest $\Upsilon$ and an face $F \in \Upsilon$, the fundamental bond is the unique bond containing $F$, and no other face from $\Upsilon$.

**Example**

$$\Upsilon = \{124, 134, 123, 135, 235\}$$

<table>
<thead>
<tr>
<th>$F$</th>
<th>$B$</th>
</tr>
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<tbody>
<tr>
<td>124</td>
<td>${124, 234}$</td>
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<td>134</td>
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<tr>
<td>123</td>
<td>${234, 123, 125}$</td>
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<tr>
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Theorem (DKM)

*For a fixed spanning forest, the collection of fundamental bonds forms a basis of cut space*
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Remark

Circuits are the circuits (minimal dependent sets) of cellular matroid.
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Bipyramid
Characteristic vectors of circuits

Fix circuit $C$

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$Flow_C(X) := (\{0\} \cup (Flow_i(X) \cap \{v : \text{supp}(v) = C\}))$ is 1-dimensional

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Topological interpretation of characteristic vector

Example

\[
\begin{pmatrix}
2 & 2 & 1 \\
0 & -2 & 2 \\
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-1 & 2 & 0
\end{pmatrix}
\]

Theorem (DKM)

\[\chi(C) = \sum_{F \in C} \pm |\tilde{H}(C \setminus F, \mathbb{Z})|\]

spans Cut \(C(X)\), where \(T\) stands for torsion part.

Definition

The characteristic vector of \(C\) is \(\chi(C)\).
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\[\tilde{H}(C \setminus F_1) = \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z};\]

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\[0 \quad -2 \quad 2\]

\[1 \quad 0 \quad -2\]

\[-1 \quad 2 \quad 0\]

\[\tilde{H}(C \setminus F_1) = \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}; \quad \chi(C) = (4, 2, 2)\]

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spans \(\text{Cut}_C(X)\), where \(T\) stands for torsion part.

Definition

The characteristic vector of \(C\) is \(\chi(C)\)
Orientation

The orientation on the faces is given by the witness for the circuit being in the kernel of the boundary. But we can still use the observation on the signs for cocircuits if, for some reason, it’s easier to get at the cocircuits than the circuits.
Fundamental circuit

Definition
Given a spanning forest $\Upsilon$ and an face $F \notin \Upsilon$, the fundamental circuit is the unique circuit in $\Upsilon \cup \{F\}$.

Example
$$\begin{align*}
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F &= \{234\} \\
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\end{align*}$$
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*For a fixed spanning forest, the collection of fundamental circuits forms a basis of flow space*