A non-partitionable Cohen-Macaulay simplicial complex

Art Duval[1], Bennet Goeckner[2], Caroline Klivans[3], Jeremy Martin[2]

[1]University of Texas at El Paso, [2]University of Kansas, [3]Brown University

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Partitionability Conjecture

Richard Stanley: "... a central combinatorial conjecture on Cohen-Macaulay complexes is the following."

Conjecture (Stanley '79; Garsia '80)

Every Cohen-Macaulay simplicial complex is partitionable.

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Stanley: "I am glad that this problem has finally been put to rest, though I would have preferred a proof rather than a counterexample. Perhaps you can withdraw your paper from the arXiv and come up with a proof instead."

Stanley depth

Definition (Stanley)

Let $S = \Bbbk[x_1, \ldots, x_n]$, and let M be a $\mathbb{Z}^n$-graded S-module. Then sdepth M denotes the Stanley depth of M.

Conjecture (Stanley '82)

$\operatorname{sdepth} M \geq \operatorname{depth} M$

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sdepth $M \geq$ depth M

Theorem (Herzog, Jahan, Yassemi '08)

If I_Δ is the Stanley-Reisner ideal of a Cohen-Macaulay complex Δ, then the inequality sdepth $S/I_\Delta \geq$ depth S/I_Δ *is equivalent to the partitionability of Δ.*

Corollary (DGKM '16)

Our counterexample disproves this conjecture as well.

Simplicial complexes

Definition (Simplicial complex)

Let V be set of vertices. Then Δ is a simplicial complex on V if:

- $\Delta \subseteq 2^V$; and
- if $\sigma \subseteq \tau \in \Delta$ implies $\tau \in \Delta$.

Higher-dimensional analogue of graph.

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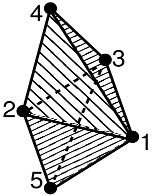
Higher-dimensional analogue of graph.

Definition (f-vector)

$f_i = f_i(\Delta) =$ number of i-dimensional faces of Δ. The f-vector of $(d-1)$-dimensional Δ is

$$f(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$$

Example



124, 125, 134, 135, 234, 235;
12, 13, 14, 15, 23, 24, 25, 34, 35;
1, 2, 3, 4, 5;
$\emptyset$

$$f(\Delta) = (1, 5, 9, 6)$$

Counting faces of spheres

Definition (Sphere)

Simplicial complex whose realization is a triangulation of a sphere.

Conjecture (Upper Bound)

Explicit upper bound on f_i *of a sphere with given dimension and number of vertices.*

This was proved by Stanley in 1975.

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Conjecture (Upper Bound)

Explicit upper bound on f_i of a sphere with given dimension and number of vertices.

This was proved by Stanley in 1975. Key ingredient:

Definition (Stanley-Reisner face-ring)

Assume Δ has vertices $1, \ldots, n$. Define $x_F = \prod_{j \in F} x_j$. Define I_Δ to be the ideal $I_\Delta = \langle x_F \colon F \notin \Delta \rangle$. The Stanley-Reisner face-ring is

$$\Bbbk[\Delta] = \Bbbk[x_1, \ldots, x_n]/I_\Delta.$$

Hilbert series

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So, for the Hilbert series,

$$F(\Bbbk[\Delta], \lambda) = \sum_{\alpha \in \mathbb{Z}^n} \dim_{\Bbbk}(\Bbbk[\Delta]_\alpha)\mathbf{t}^\alpha$$

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This means

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{k=0}^{d} h_k t^{d-k}$$

The *h-vector* of Δ is $h(\Delta) = (h_0, h_1, \ldots, h_d)$. Coefficients not always non-negative!

Cohen-Macaulay complexes

Definition (Cohen-Macaulay ring)

A ring R is Cohen-Macaulay when $\dim R = \operatorname{depth} R$.

In our setting $\dim \Bbbk[\Delta] = \dim \Bbbk[x_1, \ldots, x_n]/I_\Delta = d$.

Definition (Depth)

$(\theta_1, \ldots, \theta_r)$ is a regular sequence of module M if θ_{i+1} is non-zero divisor of $M/(\theta_1 M + \cdots + \theta_i M)$; equivalently, $\theta_1, \ldots, \theta_r$ alg. ind. over k and M is free $\Bbbk[\theta_1, \ldots, \theta_r]$-module. Then depth M is the length of the longest regular sequence of M.

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A simplicial complex Δ is Cohen-Macaulay when $\mathbb{k}[\Delta]$ is.

Remark

The h-vector is non-negative for Cohen-Macaulay complexes.

Definition (Link)

$\mathrm{lk}_\Delta\,\sigma = \{\tau \in \Delta \colon \tau \cap \sigma = \emptyset,\ \tau \cup \sigma \in \Delta\}$, what Δ looks like near σ.

Definition (Homology)

$\tilde{H}_i(\Delta) = \ker \partial_i \,/\, \mathrm{im}\,\partial_{i+1}$, measures i-dimensional "holes" of Δ.

Theorem (Reisner '76)

Face-ring of Δ is Cohen-Macaulay if, for all $\sigma \in \Delta$,

$$\tilde{H}_i(\mathrm{lk}_\Delta\,\sigma) = 0 \quad \textit{for } i < \dim \mathrm{lk}_\Delta\,\sigma.$$

Combinatorics and Topology

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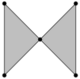
Munkres ('84) showed that CM is a topological condition. That is, it only depends on (the homeomorphism class of) the realization of Δ. In particular, spheres and balls are CM.

Example

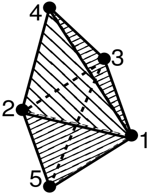
 is not CM

h-vector

Recall $\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{k=0}^{d} h_k t^{d-k}$; so

$$\sum_{i=0}^{d} f_{i-1} t^{d-i} = \sum_{k=0}^{d} h_k (t+1)^{d-k}.$$

Example



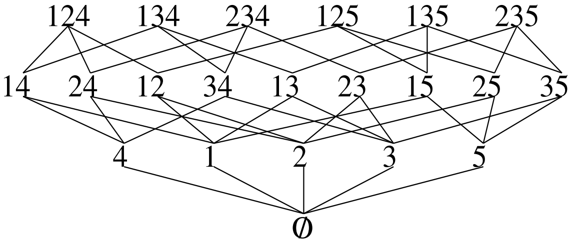
$f(\Delta) = (1, 5, 9, 6)$, and

$1t^3+5t^2+9t+6 = 1(t+1)^3+2(t+1)^2+2(t+1)^1+1$

so $h(\Delta) = (1, 2, 2, 1)$.

Partitionability

$$1t^3 + 5t^2 + 9t + 6 = \color{red}{1}(t+1)^3 + \color{blue}{2}(t+1)^2 + \color{green}{2}(t+1)^1 + 1$$



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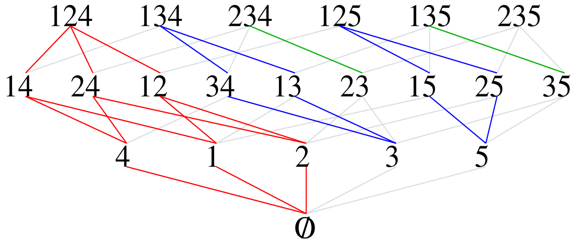

Definition (Partitionable)

When a simplicial complex can be partitioned like this, into Boolean intervals whose tops are facets, we say the complex is partitionable.

Shellability

Most CM complexes in combinatorics are shellable:

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A simplicial complex is shellable if it can be built one facet at a time, so that there is always a unique new minimal face being added.

A shelling is a particular kind of partitioning.

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Proposition

If Δ is shellable, then h_k counts number of intervals whose bottom (the unique new minimal face) is dimension $k - 1$.

Example

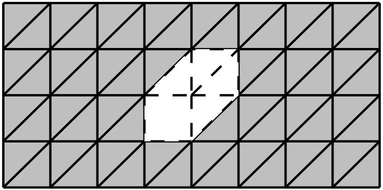
In our previous example, minimal new faces were: $\emptyset$, vertex, edge, vertex, edge, triangle.

We were trying to prove the conjecture

Idea of our "proof":

- Remove all the faces containing a given vertex (this will be the first part of the partitioning).

Example



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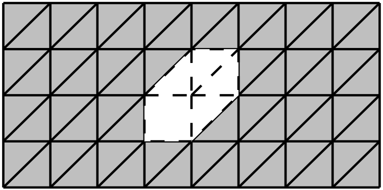

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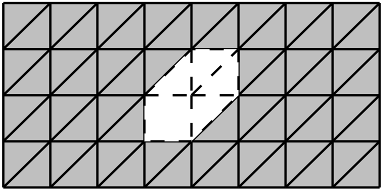

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The problem is we would have to prove the conjecture for relative CM complexes.

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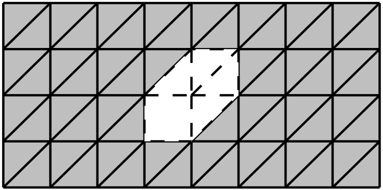

Relative simplicial complexes

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- $\Phi \subseteq 2^V$; and
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We can write any relative complex Φ as $\Phi = (\Delta, \Gamma)$, for some pair of simplicial complexes $\Gamma \subseteq \Delta$.

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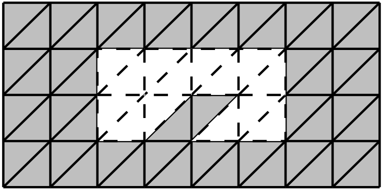

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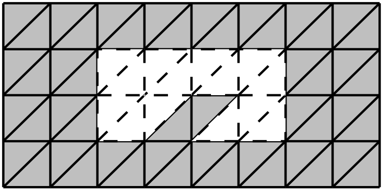

Relative Cohen-Macualay

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This generalizes easily:

Theorem (Stanley '87)

Face-ring of $\Phi = (\Delta, \Gamma)$ is relative Cohen-Macaulay if, for all $\sigma \in \Delta$,

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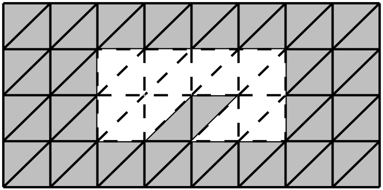

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Still trying to prove conjecture:

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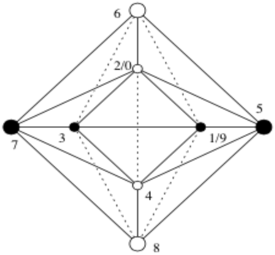
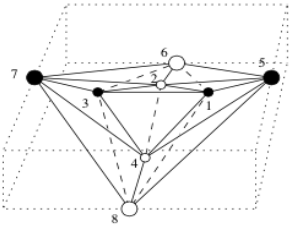
- We wanted to find a non-trivial example of something Cohen-Macaulay and partitionable, so we could see how this idea of relative complexes would work.
- How hard is it to take that second step of the partitioning, which is the first step for the relative complex?
- Idea: non-trivial = not shellable; CM = ball (and if it's not partitionable, we're done). So we are looking for non-shellable balls.

Ziegler's non-shellable ball ('98)

Non-shellable 3-ball with 10 vertices and 21 tetrahedra





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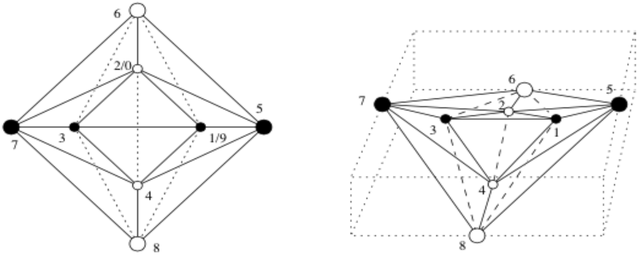
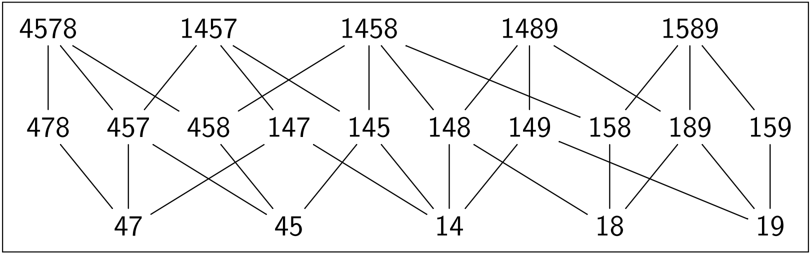


Just because it is partitionable does not mean you can start partitioning in any order.
So we started to partition until we could not go any further (without backtracking). This part uses the computer!

First pass with Ziegler

We found a relative complex $Q_5 = (X_5, A_5)$

- X_5 has 6 vertices, 5 facets



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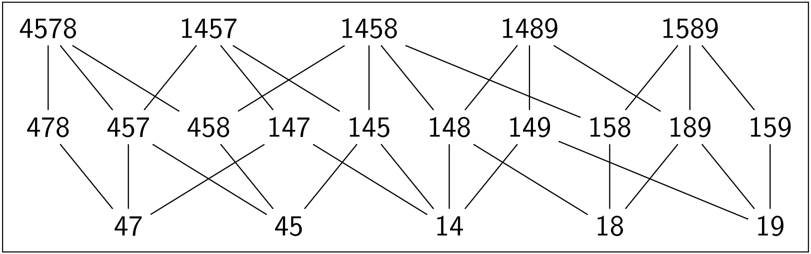

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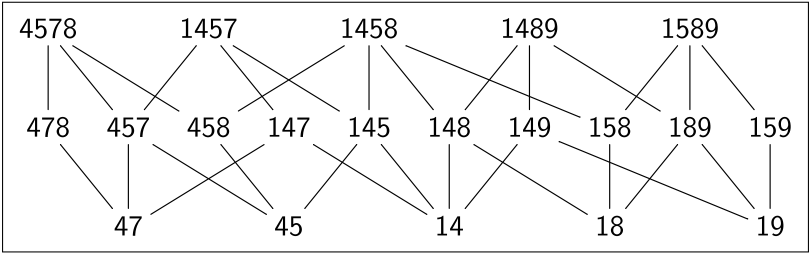

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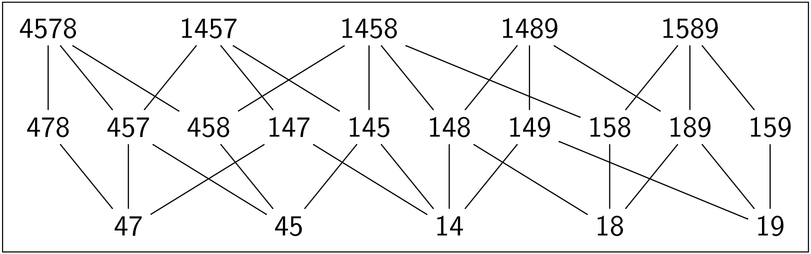

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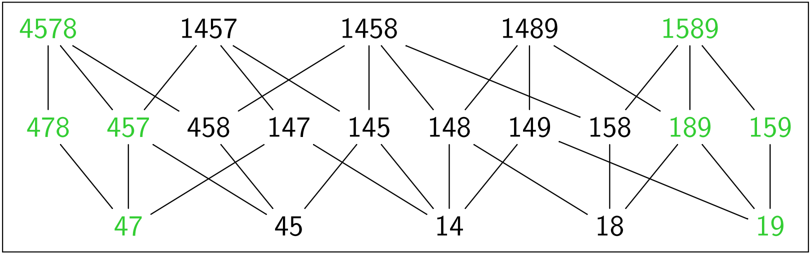

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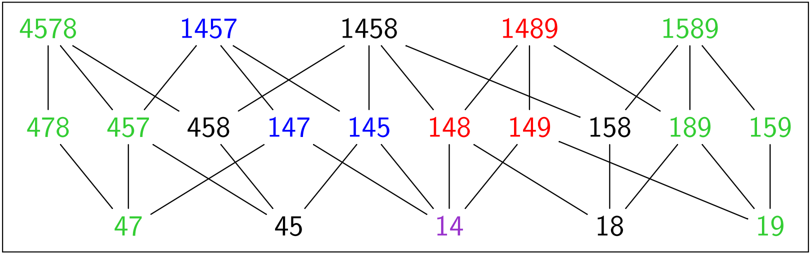

Gluing

Proposition

If X and (X, A) are CM and $\dim A = \dim X - 1$, then gluing together two copies of X along A gives a CM (non-relative) complex.

If we glue together two copies of X along A, is it partitionable?

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If we glue together two copies of X along A, is it partitionable? Maybe it is: some parts of A can help partition one copy of X, other parts of A can help partition the other copy of X.

Pigeonhole principle

Recall our example (X, A) is:

- relative Cohen-Macaulay
- not partitionable

Remark

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- relative Cohen-Macaulay
- not partitionable
- A vertex-induced (minimal faces of (X, A) are vertices)

Remark

If we glue together many copies of X along A, at least one copy will be missing all of A! How many is enough? More than the number of all faces in A. Then the result will not be partitionable.

Remark

But the resulting complex is not actually a simplicial complex because of repeats. To avoid this problem, we need to make sure that A is vertex-induced. This means every face in X among vertices in A must be in A as well. (Minimal faces of (X, A) are vertices.)

Eureka!

By computer search, we found that if

- Z is Ziegler's 3-ball, and
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Eureka!

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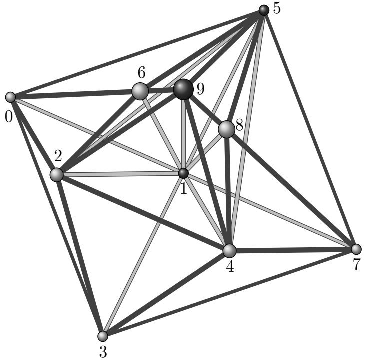
Also $Q = (X, A)$, where X has 14 facets, and A is 5 triangles:



1249	1269
1569	1589
1489	1458
1457	4578
1256	0125
0256	0123
1234	1347

Putting it all together

- Since A has 24 faces total (including the empty face), we know gluing together 25 copies of X along their common copy of A, the resulting (non-relative) complex C_{25} is CM, not partitionable.

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- In fact, computer search showed that gluing together only 3 copies of X will do it. Resulting complex C_3 has f-vector $(1, 16, 71, 98, 42)$.
- Later we found short proof by hand to show that C_3 works.

Stanley Decompositions

Definition

Let $S = \Bbbk[x_1, \ldots, x_n]$; $\mu \in S$ a monomial; and $A \subseteq \{x_1, \ldots, x_n\}$. The corresponding Stanley space in S is the vector space

$$\mu \cdot \Bbbk[A] \;=\; \Bbbk\text{-}\operatorname{span}\{\mu\nu \colon \operatorname{supp}(\nu) \subseteq A\}.$$

Let $I \subseteq S$ be a monomial ideal. A Stanley decomposition of S/I is a family of Stanley spaces

$$\mathcal{D} \;=\; \{\mu_1 \cdot \Bbbk[A_1],\ \ldots,\ \mu_r \cdot \Bbbk[A_r]\} \text{ such that}$$

$$S/I \;=\; \bigoplus_{i=1}^{r} \mu_i \cdot \Bbbk[A_i].$$

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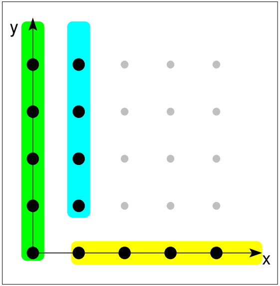
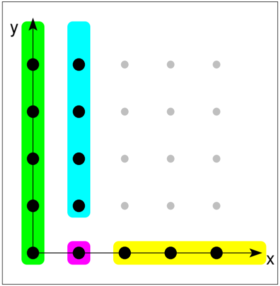
(And all of this works more generally for S-modules.)

Stanley Depth

Two Stanley decompositions of $R = \Bbbk[x,y]/\langle x^2 y\rangle$:





Definition

The Stanley depth of S/I is

$$\operatorname{sdepth} S/I \;=\; \max_{\mathcal{D}} \min\{|A_i|\}.$$

where $\mathcal{D}$ runs over all Stanley decompositions of S/I.

Depth Conjecture

Conjecture (Stanley '82)

For all monomial ideals I, $\operatorname{sdepth} S/I \geq \operatorname{depth} S/I$.

Theorem (Herzog, Jahan, Yassemi '08)

If I_Δ is the Stanley-Reisner ideal of a Cohen-Macaulay complex Δ, then the inequality $\operatorname{sdepth} S/I_\Delta \geq \operatorname{depth} S/I_\Delta$ *is equivalent to the partitionability of Δ.*

Corollary

Our counterexample disproves this conjecture as well.

Remark (Katthän)

Katthän computed (using the algorithm developed by Ichim and Zarojanu) that sdepth $C_3 = 3$ (and depth $C_3 = 4$ since it is CM).

Computations, and a new conjecture

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Conjecture (Katthän)

sdepth $\geq$ depth -1

Remark

Katthän was working on this conjecture even before our counterexample.

Constructibility

Definition

A d-dimensional simplicial complex Δ is constructible if:

- it is a simplex; or
- $\Delta = \Delta_1 \cup \Delta_2$, where $\Delta_1, \Delta_2, \Delta_1 \cap \Delta_2$ are constructible of dimensions $d, d, d-1$, respectively.

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Question (Hachimori '00)

Are constructible complexes partitionable?

Corollary

Our counterexample is constructible, so the answer to this question is no.

Open question: Smaller counterexample?

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Is the partitionability conjecture true in 2 dimensions?

Save the conjecture: Strengthen the hypothesis

More open questions (based on what our counterexample is not):
Note that our counterexample is not a ball (3 balls sharing common 2-dimensional faces), but all balls are CM.

Question

Are simplicial balls partitionable?

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Note that our counterexample is not a ball (3 balls sharing common 2-dimensional faces), but all balls are CM.

Question

Are simplicial balls partitionable?

Definition (Balanced)

A simplicial complex is balanced if vertices can be colored so that every facet has one vertex of each color.

Question

Are balanced Cohen-Macaulay complexes partitionable?

Question

What does the h-vector of a CM complex count?

Save the conjecture: Weaken the conclusion

Question

What *does* *the h-vector of a CM complex count?*

One possible answer (D.-Zhang '01) replaces Boolean intervals with "Boolean trees". But maybe there are other answers.