The $G$-Shi arrangement, and its relation to $G$-parking functions

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Joint Mathematics Meetings
New Orleans, LA
January 8, 2011
**Arrangements**

Braid $B_n := \{x_i = x_j : 1 \leq i < j \leq n\}$ \(n!\) regions
Arrangements

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Shi $S_n := \{x_i = x_j, x_i = x_j + 1: 1 \leq i < j \leq n\}$ $(n + 1)^{n-1}$ regions
Arrangements

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Shi $S_n := \{x_i = x_j, x_i = x_j + 1 : 1 \leq i < j \leq n\}$ \((n + 1)^{n-1}\) regions

\((n + 1)^{n-1}\) is also the number of spanning trees of $K_n$ (Cayley)
Parking functions

Definition

- parking spots 0, \ldots, n - 1

Example

\begin{tabular}{cccc}
3 & 2 & 1 & 0 \\
\end{tabular}
Parking functions

Definition

- parking spots 0, \ldots, n - 1
- cars 1, \ldots, n arrive in order

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Parking functions

Definition

▶ parking spots $0, \ldots, n - 1$
▶ cars $1, \ldots, n$ arrive in order
▶ car $i$ has favorite parking spot $f(i)$

Example

1120

3 2 1 0
Parking functions

Definition

- parking spots 0, \ldots, n - 1
- cars 1, \ldots, n arrive in order
- car \( i \) has favorite parking spot \( f(i) \)
- car \( i \) goes first to spot \( f(i) \) \ldots

Example

1120

\[
\begin{array}{c|c|c|c}
3 & 2 & 1 & 0 \\
\end{array}
\]

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G-Shi arrangement and G-parking functions
Parking functions

Definition

- parking spots $0, \ldots, n - 1$
- cars $1, \ldots, n$ arrive in order
- car $i$ has favorite parking spot $f(i)$
- car $i$ goes first to spot $f(i)$ . . .
- . . . if that spot is full, takes next available spot

Example

1120

```

3

2 1 1

1 1

0

```

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- parking spots 0, . . . , n − 1
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Example

1120

\[ \begin{array}{c|c|c|c}
3 & 2 & 1 & 1 \\
\hline
2 & 1 & 1 & 0 \\
\end{array} \]
Parking functions

Definition

- parking spots 0, . . . , $n - 1$
- cars 1, . . . , $n$ arrive in order
- car $i$ has favorite parking spot $f(i)$
- car $i$ goes first to spot $f(i)$ . . .
- . . . if that spot is full, takes next available spot

If such a function $f$ allows all the cars to park, it is a parking function. [Note that indexing is sometimes different.]

Example

1120

\[
\begin{array}{c}
3 & 2 & 2 & 1 & 1 & 0 & 0 \\
\end{array}
\]
Which functions are parking functions?

Example
Easy: All 0’s;

\[
\begin{array}{cccc}
3 & 2 & 1 & 0 \\
\end{array}
\]
Which functions are parking functions?

Example
Easy: All 0’s; any permutation of 0, . . . , $n – 1$. 
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Example
These are not parking functions: 3003,
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These are not parking functions: 3003, 2230, 1121
Which functions are parking functions?

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Easy: All 0’s; any permutation of 0, ..., \( n - 1 \).

Example
These are not parking functions: 3003, 2230, 1121
Necessary: Fewer than \( i \) cars whose value is greater than \( n - i \)
Which functions are parking functions?

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Easy: All 0’s; any permutation of 0, . . . , n − 1.

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These are not parking functions: 3003, 2230, 1121
Necessary: Fewer than \( i \) cars whose value is greater than \( n - i \)
Equivalently, when values \( f(i) \) rearranged in increasing order, \( f(i) < i \). (\( f \) is “componentwise” less than permutation of 0, . . . , \( n - 1 \)).
Which functions are parking functions?

Example
Easy: All 0’s; any permutation of 0, ..., n – 1.

Example
These are not parking functions: 3003, 2230, 1121
Necessary: Fewer than i cars whose value is greater than n – i
Equivalently, when values f(i) rearranged in increasing order, f(i) < i. (f is “componentwise” less than permutation of 0, ..., n – 1.)
This is sufficient, too (making values less only makes it easier to park).

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G-Shi arrangement and G-parking functions
How many are there?

\[ n=2: \ 00, \ 01, \ 10 \]
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\[ n=2: \ 00, \ 01, \ 10 \]
\[ n=3: \ 000, \ 001, \ 010, \ 100, \ 011, \ 101, \ 110, \ 002, \ 020, \ 200, \ 012, \ 021, \ 102, \ 120, \ 201, \ 210 \]
How many are there?

\[ n=2: \ 00,\ 01,\ 10 \]
\[ n=3: \ 000,\ 001,\ 010,\ 100,\ 011,\ 101,\ 110,\ 002,\ 020,\ 200,\ 012,\ 021,\ 102,\ 120,\ 201,\ 210 \]

Theorem (Pyke, '59; Konheim and Weis, '66)

There are \((n + 1)^{n-1}\) parking functions.
Pak-Stanley labelling

Pak and Stanley found a labelling of the regions of the Shi arrangement so that each region gets a different label,
Pak-Stanley labelling

Pak and Stanley found a labelling of the regions of the Shi arrangement so that each region gets a different label, and each label is a parking function!

Athanasiadis and Linusson have alternate (easier) bijection between parking functions and Shi regions.
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Athanasiadis and Linusson have alternate (easier) bijection between parking functions and Shi regions.

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G-Shi arrangement and G-parking functions
Restating parking function definition

Recall the original necessary and sufficient condition:

More than $i$ cars whose value is greater than $n - i$.

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G-Shi arrangement and G-parking functions
Restating parking function definition

Recall the original necessary and sufficient condition:

Fewer than \( i \) cars whose value is greater than \( n - i \).

Restate this as:

In any set of \( i \) cars, there is at least one whose value is at most \( n - i \).
Restating parking function definition

Recall the original necessary and sufficient condition:

Fewer than \( i \) cars whose value is greater than \( n - i \).

Restate this as:

In any set of \( i \) cars, there is at least one whose value is at most \( n - i \).
G-parking functions

Definition (Postnikov-Shapiro ’04)
Given a graph $G = (V, E)$, a function $f : V \rightarrow \mathbb{Z}_{\geq 0}$ is a parking function if, in any set $U \subseteq V$ of vertices, there is at least one vertex $v$ such that $f(v)$ is at most the $\bar{U}$-degree of $v$, the number of neighbors of $v$ outside of $U$. 

![Diagram of a graph with parking function example]
**G-parking functions**

**Definition (Postnikov-Shapiro ’04)**

Given a graph $G = (V, E)$, with root $q$, a function $f : V \setminus q \rightarrow \mathbb{Z}^+ \geq 0$ is a **parking function** if, in any set $U \subseteq V \setminus q$ of vertices, there is at least one vertex $v$ such that $f(v)$ is at most the $\bar{U}$-degree of $v$, the number of neighbors of $v$ outside of $U$.

Note that if $G = K_{n+1}$ we get classical parking functions on $n$ cars.
Spanning Trees

Theorem (Postnikov-Shapiro)

\[ \# \{ G\text{-parking functions} \} = \# \{ \text{spanning trees of } G \ast 0 \}. \]
Graphical arrangement

Start with braid arrangement, but include only hyperplanes corresponding to edges in graph:

$$\{x_i = x_j : i < j; \{i, j\} \in E\}$$
Graphical arrangement

Start with braid arrangement, but include only hyperplanes corresponding to edges in graph:

\[ \{ x_i = x_j : i < j; \{i,j\} \in E \} \]

Example

\[
\begin{align*}
1 & \quad 2 & \quad 3 \\
x_2 &= x_3 & x_1 &= x_2
\end{align*}
\]
**G-Shi arrangement**

If we combine the ideas of the graphical arrangement and the Shi arrangement, we get

\[
\{ x_i = x_j, x_i = x_j + 1 : i < j; \{i, j\} \in E \}
\]
G-Shi arrangement

If we combine the ideas of the graphical arrangement and the Shi arrangement, we get

\[ \{ x_i = x_j, x_i = x_j + 1 : i < j; \{i, j\} \in E \} \]

But this has 9 regions, and there are only 8 spanning trees and 8 parking functions.
Conjecture

There is a bijection between the $G^*$-parking functions and the set of different labels of the $G$-Shi arrangement.
Conjecture

There is a bijection between the \( G_0 \)-parking functions and the set of different labels of the \( G \)-Shi arrangement.

\[
\begin{align*}
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 2 & 0
\end{array}
\end{align*}
\]
Conjecture

There is a bijection between the $G^*$-parking functions and the set of different labels of the $G$-Shi arrangement.
Conjecture

There is a bijection between the \((G \ast 0)\)-parking functions and the set of different labels of the \(G\)-Shi arrangement.
Maximal labels in $G$-Shi

What are the maximal labels (maximum total weight) of the $G$-Shi arrangement?
Maximal labels in $G$-Shi

What are the maximal labels (maximum total weight) of the $G$-Shi arrangement?
The regions can’t be in any of the “middle slices”

\[
\begin{align*}
&x_1 = x_2 + 1 \\
&x_2 = x_3 + 1 \\
&x_1 = x_2 \\
&x_2 = x_3
\end{align*}
\]
Maximal labels in $G$-Shi

What are the maximal labels (maximum total weight) of the $G$-Shi arrangement?
The regions can’t be in any of the “middle slices”

\[
x_2 = x_3
\]
\[
x_2 = x_3 + 1
\]
\[
x_1 = x_2 + 1
\]
\[
x_1 = x_2
\]
Graphical arrangement

For every pair of parallel hyperplanes (corresponding to edge in graph), you have to be on one side or the other:

\[ 010011 \]
\[ 010 \]
\[ 100 \]
\[ 000001 \]
\[ 101 \]
\[ 020 \]
\[ x_1 = x_2 \]
\[ x_1 = x_2 + 1 \]

\[ 101 \]
\[ 100 \]
\[ 110 \]

\[ x_2 = x_3 \]
\[ x_2 = x_3 + 1 \]

\[ 001 \]
\[ 000 \]
\[ 010 \]
\[ x_1 = x_2 + 1 \]

\[ 011 \]
\[ 010 \]
\[ 020 \]

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G-Shi arrangement and G-parking functions
Graphical arrangement

For every pair of parallel hyperplanes (corresponding to edge in graph), you have to be on one side or the other: Graphical arrangement!
Graphical arrangement

For every pair of parallel hyperplanes (corresponding to edge in graph), you have to be on one side or the other: Graphical arrangement!
Weight goes up by one for every hyperplane crossed, so total weight is number of edges of $G$. 

$\begin{align*}
010011 &\quad 010 \\
100 &\quad 000001 \\
101 &\quad 020 \quad x_1 = x_2 + 1 \\
011 &\quad 010 \\
010 &\quad 020 \\
&\quad x_1 = x_2 \\
&\quad x_2 = x_3 + 1 \\
&\quad x_2 = x_3 \\
\end{align*}$
Regions of graphical arrangement correspond to acyclic orientations on graph (just like regions of braid arrangement correspond to permutations, which correspond to acyclic orientations of the complete graph).

So there is a natural bijection between maximal labels of the $G$-Shi arrangement and acyclic orientations of $G$. 

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G-Shi arrangement and G-parking functions
Example: $K_n$ again

\begin{align*}
x_2 &= x_3 \\
x_2 &= x_3 + 1 \\
x_1 &= x_2 + 1 \\
x_1 &= x_2 \\
x_1 &= x_3 + 1 \\
x_1 &= x_3
\end{align*}
Example: $K_n$ again
Maximal $G$-parking functions

Theorem (Benson, Chakrabarty, Tetali, ’10)

Maximal $(G \ast 0)$-parking functions also have weight equal to the number of edges of $G$, and correspond to acyclic orientations of $G$. 
Maximal $G$-parking functions

**Theorem (Benson, Chakrabarty, Tetali, ’10)**

Maximal $(G \ast 0)$-parking functions also have weight equal to the number of edges of $G$, and correspond to acyclic orientations of $G$.

**Observation (Easy)**

If $f$ is a $G$-parking function, and $g(v) \leq f(v)$ for all $v$, then $g$ is also a $G$-parking function

**Proof.**

Reducing the values of the parking function can only make it easier to satisfy the condition.

□
Maximal $G$-parking functions

Theorem (Benson, Chakrabarty, Tetali, ’10)

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Observation (Easy)

If $f$ is a $G$-parking function, and $g(v) \leq f(v)$ for all $v$, then $g$ is also a $G$-parking function

Proof.

Reducing the values of the parking function can only make it easier to satisfy the condition.

Consequence: If we could only show that labels also satisfy the easy observation, we’d be done.
Half the bijection

We can use this to easily show that every label $g$ has a corresponding parking function: There exists some maximal label $f$ such that $g(v) \leq f(v)$ for all $v$ ($g = f$ is possible). Since $f$ is maximal, it corresponds to an acyclic orientation $O$. By BCT, we know $O$ corresponds to a maximal parking function, so $f$ is a maximal parking function. By the easy observation, $g$ is also a parking function.
What about the other half?

We still need to show either [equivalently]:

- Every parking function is a label
- Labels satisfy the easy observation