Weighted spanning tree enumerators of complete colorful complexes

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Spanning trees of K_n

Theorem (Cayley)

K_n has n^{n-2} spanning trees.

$T \subseteq E(G)$ is a **spanning tree** of G when:

0. spanning: T contains all vertices;
1. connected ($\tilde{H}_0(T) = 0$)
2. no cycles ($\tilde{H}_1(T) = 0$)
3. correct count: $|T| = n - 1$

If 0. holds, then any two of 1., 2., 3. together imply the third condition.

Theorem (Cayley-Prüfer)

$$\sum_{T \in ST(K_n)} \operatorname{wt} T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2},$$

where $\operatorname{wt} T = \prod_{e \in T} \operatorname{wt} e = \prod_{e \in T} (\prod_{v \in e} x_v)$.

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Example (K_4)

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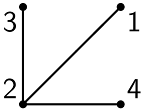
Example (K_4)

- 4 trees like: $T =$  $\text{wt } T = (x_1 x_2 x_3 x_4) x_2^2$

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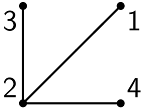
- 4 trees like: $T =$ 

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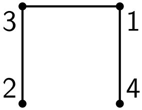
- 12 trees like: $T =$ 

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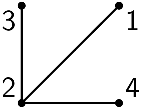
- 4 trees like: $T =$ 

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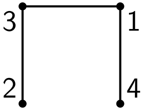
- 12 trees like: $T =$ 

$\operatorname{wt} T = (x_1 x_2 x_3 x_4) x_1 x_3$
- Total is $(x_1 x_2 x_3 x_4)(x_1 + x_2 + x_3 + x_4)^2$.

Complete bipartite graphs

Example ($K_{3,2}$)

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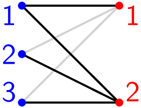
- 6 trees like: $T =$  $\operatorname{wt} T = (12312)12^2$

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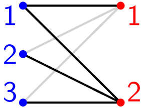
- 6 trees like: $T =$ 

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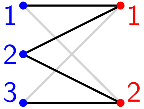
- 6 trees like: $T =$ 

wt $T = (12312)212$

Complete bipartite graphs

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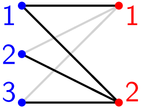
- 6 trees like: $T =$ 

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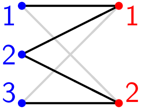
- 6 trees like: $T =$ 

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- Total is $(12312)(1+2+3)(1+2)^2$.

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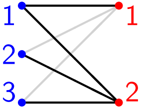
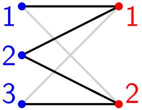
- 6 trees like: $T =$  wt $T = (12312)12^2$
- 6 trees like: $T =$  wt $T = (12312)212$
- Total is $(12312)(1+2+3)(1+2)^2$.

Theorem

$$\sum_{T \in ST(K_{m,n})} \operatorname{wt} T = (x_1 \cdots x_m)(y_1 \cdots y_n)(x_1 + \cdots + x_m)^{n-1}(y_1 + \cdots + y_n)^{m-1}.$$

Laplacian

Theorem (Kirchoff's Matrix-Tree)

G has $|\det L_r(G)|$ spanning trees

Definition The Laplacian matrix of graph G, denoted by $L(G)$.

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Defn 1: $L(G) = D(G) - A(G)$

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$$A(G) = \text{adjacency matrix}$$

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$$\partial(G) = \text{incidence matrix (boundary matrix)}$$

Laplacian

Theorem (Kirchoff's Matrix-Tree)

G has $|\det L_r(G)|$ spanning trees

Definition The reduced Laplacian matrix of graph G, denoted by $L_r(G)$.

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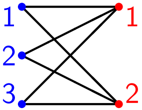
$$\partial(G) = \text{incidence matrix (boundary matrix)}$$

"Reduced": remove rows/columns corresponding to any one vertex

Example ($K_{3,2}$)



$$\partial = \begin{array}{c|cccccc} & 11 & 12 & 21 & 22 & 31 & 32 \\ \hline 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 & -1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

$$L = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 2 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & -1 & 0 & 3 \end{pmatrix}$$

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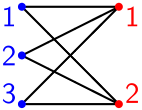

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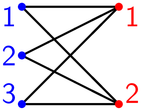

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$\det(L_r) = 12$, the number of spanning trees of $K_{3,2}$.

Weighted Matrix-Tree Theorem

$$\sum_{T \in ST(G)} \mathrm{wt}\, T = |\det \hat{L}_r(G)|,$$

where $\hat{L}_r(G)$ is reduced weighted Laplacian.

Defn 1: $\hat{L}(G) = \hat{D}(G) - \hat{A}(G)$

$\hat{D}(G) = \mathrm{diag}(\hat{\deg} v_1, \ldots, \hat{\deg} v_n)$

$\hat{\deg} v_i = \sum_{v_i v_j \in E} x_i x_j$

$\hat{A}(G) =$ adjacency matrix

(entry $x_i x_j$ for edge $v_i v_j$)

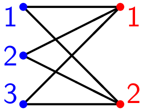
Defn 2: $\hat{L}(G) = \partial(G) B(G) \partial(G)^T$

$\partial(G) =$ incidence matrix

$B(G)$ diagonal, indexed by edges,

entry $\pm x_i x_j$ for edge $v_i v_j$

Example ($K_{3,2}$)



$$\hat{L}_r = \begin{pmatrix} 2(1+2) & 0 & -21 & -22 \\ 0 & 3(1+2) & -31 & -32 \\ -21 & -31 & 1(1+2+3) & 0 \\ -22 & -32 & 0 & 2(1+2+3) \end{pmatrix}$$

$$\det \hat{L}_r = (12312)(1+2+3)(1+2)^2$$

Proof of $K_{3,2}$ formula

$$\det \begin{pmatrix} 2(1+2) & 0 & -21 & -22 \\ 0 & 3(1+2) & -31 & -32 \\ -21 & -31 & 1(1+2+3) & 0 \\ -22 & -32 & 0 & 2(1+2+3) \end{pmatrix}$$

$$2312 \det \begin{pmatrix} 1+2 & 0 & -1 & -2 \\ 0 & 1+2 & -1 & -2 \\ -2 & -3 & 1+2+3 & 0 \\ -2 & -3 & 0 & 1+2+3 \end{pmatrix}$$

By “identification of factors” (Martin-Reiner, ’03), to show $(1+2)^2$ is a factor of the determinant, we just have to show that the nullspace of this matrix is at least 2, when we set $1+2=0$.

Finding null vectors

$$\begin{pmatrix} 1+2 & 0 & -1 & -2 \\ 0 & 1+2 & -1 & -2 \\ -2 & -3 & 1+2+3 & 0 \\ -2 & -3 & 0 & 1+2+3 \end{pmatrix}$$

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Since we removed 2 more rows than columns, nullity is at least 2.

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Since we removed 2 more rows than columns, nullity is at least 2. Any null vector (a, b, c) of 1×3 matrix gives null vector (a, b, c, c) of 4×4 matrix. (Remember $1 + 2 = 0$.)

Finding null vectors

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Since we removed 2 more rows than columns, nullity is at least 2. Any null vector (a, b, c) of 1×3 matrix gives null vector (a, b, c, c) of 4×4 matrix. (Remember $1 + 2 = 0$.)
We now have factors $12(1+2)^2$. To get the blue factors, now pick 1 as the vertex to be removed!

Simplicial spanning trees of K_n^d [Kalai, '83]

Let K_n^d denote the complete d-dimensional simplicial complex on n vertices. $\Upsilon \subseteq K_n^d$ is a **simplicial spanning tree** of K_n^d when:

0. $\Upsilon_{(d-1)} = K_n^{d-1}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $|\Upsilon| = \binom{n-1}{d}$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.

Counting simplicial spanning trees of K_n^d

Conjecture [Bolker '76]

$$\sum_{\Upsilon \in SST(K_n^d)} \qquad = n^{\binom{n-2}{d}}$$

Counting simplicial spanning trees of K_n^d

Theorem [Kalai '83]

$$\tau(K_n^d) = \sum_{\Upsilon \in SST(K_n^d)} |\tilde{H}_{d-1}(\Upsilon)|^2 = n^{\binom{n-2}{d}}$$

Weighted simplicial spanning trees of K_n^d

As before,

$$\operatorname{wt}\Upsilon = \prod_{F\in\Upsilon} \operatorname{wt} F = \prod_{F\in\Upsilon} \Big(\prod_{v\in F} x_v\Big)$$

Example

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$
$$\operatorname{wt}\Upsilon = x_1^5 x_2^4 x_3^3 x_4^3 x_5^3$$

Weighted simplicial spanning trees of K_n^d

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Example

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$
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Theorem (Kalai, '83)

$$\hat{\tau}(K_n^d) := \sum_{T\in SST(K_n^d)} |\tilde{H}_{d-1}(\Upsilon)|^2 (\mathrm{wt}\,\Upsilon)$$
$$= (x_1\cdots x_n)^{\binom{n-2}{d-1}} (x_1+\cdots+x_n)^{\binom{n-2}{d}}$$

Proof

Proof uses determinant of reduced Laplacian of K_n^d. "Reduced" now means pick one vertex, and then remove rows/columns corresponding to all $(d-1)$-dimensional faces containing that vertex.
$L = \partial\partial^T$
$\partial \colon \Delta_d \to \Delta_{d-1}$ boundary
$\partial^T \colon \Delta_{d-1} \to \Delta_d$ coboundary
Weighted version: Multiply column F of ∂ by x_F

Example $n = 4, d = 2$ (tetrahedron)

$$\partial^T = \begin{array}{c|cccccc} & 12 & 13 & 14 & 23 & 24 & 34 \\ \hline 123 & -1 & 1 & 0 & -1 & 0 & 0 \\ 124 & -1 & 0 & 1 & 0 & -1 & 0 \\ 134 & 0 & -1 & 1 & 0 & 0 & -1 \\ 234 & 0 & 0 & 0 & -1 & 1 & -1 \end{array}$$

$$L = \begin{pmatrix} 2 & -1 & -1 & 1 & 1 & 0 \\ -1 & 2 & -1 & -1 & 0 & 1 \\ -1 & -1 & 2 & 0 & -1 & -1 \\ 1 & -1 & 0 & 2 & -1 & 1 \\ 1 & 0 & -1 & -1 & 2 & -1 \\ 0 & 1 & -1 & 1 & -1 & 2 \end{pmatrix}$$

$det L_r = 4$

Simplicial spanning trees of arbitrary simplicial complexes

Let Δ be a d-dimensional simplicial complex.
$\Upsilon \subseteq \Delta$ is a **simplicial spanning tree** of Δ when:

0. $\Upsilon_{(d-1)} = \Delta_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.

Simplicial Matrix-Tree Theorem

Theorem (D.-Klivans-Martin, '09)

$$\hat{\tau}(\Delta) = \frac{|\tilde{H}_{d-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det \hat{L}_\Gamma,$$

where

- $\Gamma \in SST(\Delta_{(d-1)})$
- ∂_Γ = *restriction of* ∂_d *to faces not in* Γ
- *reduced Laplacian* $L_\Gamma = \partial_\Gamma {\partial^T}_\Gamma$
- *Weighted version: Multiply column* F *of* ∂ *by* x_F

Note: The $|\tilde{H}_{d-2}|$ terms are often trivial.

Example: Octahedron

- Vertices 1, 2, 1, 2, 1, 2.
- Facets 111, 112, 121, 122, 211, 212, 221, 222,
- $\Gamma = 11, 12, 11, 12, 22$ spanning tree of 1-skeleton, so remove (rows and columns corresponding to) those edges from weighted Laplacian.
- $\det \hat{L}_\Gamma = (121212)^3(1+2)(1+2)(1+2)$.

Complete colorful complexes

Definition (Adin, '92)

The complete colorful complex $K_{n_1,\ldots,n_r}$ is a simplicial complex with:

- vertex set $V_1 \dot{\cup} \ldots \dot{\cup} V_r$ (V_i is set of vertices of color i);
- $|V_i| = n_i$;
- faces are all sets of vertices with no repeated colors.

Example

Octahedron is K_{222}.

Unweighted enumeration

Theorem (Adin, '92)

The top-dimensional spanning trees of $K_{n_1,\dots,n_r}$ are "counted" by

$$\tau(K_{n_1,\dots,n_r}) = \prod_{i=1}^{r} n_i^{\prod_{j\neq i}(n_j-1)}.$$

Note: Adin also has a more general formula for dimension less than $r-1$.

Example

- $\tau(K_{222}) = 2^1 \times 2^1 \times 2^1$
- $\tau(K_{235}) = 2^{2\cdot 4} \times 3^{1\cdot 4} \times 5^{1\cdot 2}$
- $\tau(K_{m,n}) = m^{n-1} \times n^{m-1}$

Weighted enumeration

Theorem (Aalipour-D.)

The top-dimensional spanning trees of $K_{n_1,\dots,n_r}$ are "counted" by $\tau(K_{n_1,\dots,n_r}) =$

$$\prod_{i=1}^{r} (x_{i,1} + \cdots + x_{i,n_i})^{\prod_{j\neq i}(n_j-1)} (x_{i,1}\cdots x_{i,n_i})^{(\prod_{j\neq i} n_j)-(\prod_{j\neq i}(n_j-1))}.$$

Example

$$\begin{aligned}\hat{\tau}(K_{235}) = {} & (x_1+x_2)^{2\cdot 4}(x_1x_2)^{3\cdot 5-2\cdot 4} \\ & \times (y_1+y_2+y_3)^{1\cdot 4}(y_1y_2y_3)^{2\cdot 5-1\cdot 4} \\ & \times (z_1+\cdots+z_5)^{1\cdot 2}(z_1\cdots z_5)^{2\cdot 3-1\cdot 2}\end{aligned}$$

Codimension-1 spanning tree (Adin)

We will use the weighted simplicial matrix-tree theorem. So first we have to find a codimension-1 spanning tree. But it will be a different tree for each color. For each color's factors, treat that color as "last".

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$r = 3$ (1-dimensional spanning tree): Start with 1, and attach to every other vertex, except blue vertices. Then use 1 to connect the remaining blue vertices.

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$r = 3$ (1-dimensional spanning tree): Start with 1, and attach to every other vertex, except blue vertices. Then use 1 to connect the remaining blue vertices.

$r = 4$ (2-dimensional spanning tree): Start with 1, and attach to every edge with no blue vertices. Then use 1, and attach to all edges using a blue non-1 vertex with a non-red vertex. Finally use 1 with edges with a blue non-1 vertex with a red non-1 vertex.

Continuing proof

The rest of the proof is similar to our $K_{3,2}$ computation:

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- Now apply identification of factors:
  - remove the rows containing variables of the last color (number of rows is degree of sum of variables of this color)
  - remove "duplicate" rows and columns
  - null vectors of resulting matrix can be expanded to null vectors of full reduced matrix.

Final thought

Terry Pratchett, *The Colour of Magic*:
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But, now, *you* do.