Spanning trees of graphs
Spanning trees of simplicial complexes
Complete colorful complexes

Weighted spanning tree enumerators of complete colorful complexes

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Spanning trees of $K_n$

**Theorem (Cayley)**

$K_n$ has $n^{n-2}$ spanning trees.

$T \subseteq E(G)$ is a **spanning tree** of $G$ when:

1. **spanning**: $T$ contains all vertices;
2. **connected** ($\tilde{H}_0(T) = 0$)
3. **no cycles** ($\tilde{H}_1(T) = 0$)
4. **correct count**: $|T| = n - 1$

If 0. holds, then any two of 1., 2., 3. together imply the third condition.
Theorem (Cayley-Prüfer)

\[
\sum_{T \in ST(K_n)} \text{wt } T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2},
\]

where \( \text{wt } T = \prod_{e \in T} \text{wt } e = \prod_{e \in T} (\prod_{v \in e} x_v) \).
Theorem (Cayley-Prüfer)

\[
\sum_{T \in ST(K_n)} wt \ T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2},
\]

where \( wt \ T = \prod_{e \in T} wt \ e = \prod_{e \in T} (\prod_{v \in e} x_v) \).

Example \((K_4)\)
Theorem (Cayley-Prüfer)

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where \( \text{wt } T = \prod_{e \in T} \text{wt } e = \prod_{e \in T} (\prod_{v \in e} x_v). \)

Example \((K_4)\)

\[\begin{align*}
\text{4 trees like: } & T = \\
\text{wt } T = (x_1 x_2 x_3 x_4) x_2^2
\end{align*}\]
Theorem (Cayley-Prüfer)

\[
\sum_{T \in ST(K_n)} \text{wt } T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2},
\]

where \( \text{wt } T = \prod_{e \in T} \text{wt } e = \prod_{e \in T} (\prod_{v \in e} x_v) \).

Example \((K_4)\)

- 4 trees like: \( T = \)

\[
\begin{array}{c}
3 \\
2 \\
3 \\
1
\end{array}
\]

\( \text{wt } T = (x_1x_2x_3x_4)x_2^2 \)

- 12 trees like: \( T = \)

\[
\begin{array}{c}
3 \\
2 \\
1 \\
4
\end{array}
\]

\( \text{wt } T = (x_1x_2x_3x_4)x_1x_3 \)
Theorem (Cayley-Prüfer)

\[
\sum_{T \in ST(K_n)} \text{wt} \ T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2},
\]

where \( \text{wt} \ T = \prod_{e \in T} \text{wt} e = \prod_{e \in T} (\prod_{v \in e} x_v) \).

Example (\( K_4 \))

- 4 trees like: \( T = \)
  \[
  \begin{array}{c}
  3 \\
  2 \\
  3 \\
  1 \\
  4 \\
  \end{array}
  \]
  \( \text{wt} \ T = (x_1x_2x_3x_4)x_2^2 \)

- 12 trees like: \( T = \)
  \[
  \begin{array}{c}
  2 \\
  4 \\
  3 \\
  1 \\
  \end{array}
  \]
  \( \text{wt} \ T = (x_1x_2x_3x_4)x_1x_3 \)

- Total is \((x_1x_2x_3x_4)(x_1 + x_2 + x_3 + x_4)^2 \).
Complete bipartite graphs

Example \((K_{3,2})\)
Complete bipartite graphs

Example \((K_{3,2})\)

\[
\begin{align*}
\text{6 trees like: } T &= \begin{array}{c}
\text{1} \\
\text{2} \\
\text{3}
\end{array} \quad \begin{array}{c}
\text{1} \\
\text{2} \\
\text{2}
\end{array} \\
\text{wt } T &= (12312)12^2
\end{align*}
\]
Complete bipartite graphs

Example \((K_{3,2})\)

- 6 trees like: \(T = \)

\[
\begin{array}{c}
1 \\
2 \\
3 \\
1 \\
2 \\
3
\end{array}
\]

\(\text{wt } T = (12312)12^2\)

- 6 trees like: \(T = \)

\[
\begin{array}{c}
1 \\
2 \\
3 \\
1 \\
2 \\
3
\end{array}
\]

\(\text{wt } T = (12312)212\)
Complete bipartite graphs

Example \((K_{3,2})\)

- 6 trees like: \(T =\) wt \(T = (12312)12^2\)
- 6 trees like: \(T =\) wt \(T = (12312)212\)
- Total is \((12312)(1 + 2 + 3)(1 + 2)^2\).
Complete bipartite graphs

Example \((K_{3,2})\)

- 6 trees like: \(T = \begin{array}{c}
1 \\
2 \\
3 \\
\end{array} \begin{array}{c}
1 \\
2 \\
3 \\
\end{array} \begin{array}{c}
1 \\
2 \\
3 \\
\end{array} \begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\)
  \[\text{wt } T = (12312)12^2\]

- 6 trees like: \(T = \begin{array}{c}
1 \\
2 \\
3 \\
\end{array} \begin{array}{c}
1 \\
2 \\
3 \\
\end{array} \begin{array}{c}
1 \\
2 \\
3 \\
\end{array} \begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\)
  \[\text{wt } T = (12312)212\]

- Total is \((12312)(1 + 2 + 3)(1 + 2)^2\).

**Theorem**

\[
\sum_{T \in ST(K_{m,n})} \text{wt } T = (x_1 \cdots x_m)(y_1 \cdots y_n)(x_1 + \cdots + x_m)^{n-1}(y_1 + \cdots + y_n)^{m-1}.
\]
Laplacian

Theorem (Kirchoff’s Matrix-Tree)

\( G \) has \( |\det L_r(G)| \) spanning trees

Definition The Laplacian matrix of graph \( G \), denoted by \( L(G) \).
Laplacian

Theorem (Kirchoff’s Matrix-Tree)

\(G\) has \(|\det L_r(G)|\) spanning trees

Definition The Laplacian matrix of graph \(G\), denoted by \(L(G)\).

Defn 1: \(L(G) = D(G) - A(G)\)

\(D(G) = \text{diag}(\deg v_1, \ldots, \deg v_n)\)

\(A(G) = \text{adjacency matrix}\)
Laplacian

Theorem (Kirchoff’s Matrix-Tree)

$G$ has $|\det L_r(G)|$ spanning trees

Definition The Laplacian matrix of graph $G$, denoted by $L(G)$.

Defn 1: $L(G) = D(G) - A(G)$

$D(G) = \text{diag}(\deg v_1, \ldots, \deg v_n)$

$A(G) = \text{adjacency matrix}$

Defn 2: $L(G) = \partial(G)\partial(G)^T$

$\partial(G) = \text{incidence matrix (boundary matrix)}$
Laplacian

**Theorem (Kirchoff’s Matrix-Tree)**

$G$ has $| \det L_r(G) |$ spanning trees

**Definition** The reduced Laplacian matrix of graph $G$, denoted by $L_r(G)$.

**Defn 1:** $L(G) = D(G) - A(G)$

$D(G) = \text{diag}(\deg v_1, \ldots, \deg v_n)$

$A(G) = \text{adjacency matrix}$

**Defn 2:** $L(G) = \partial(G)\partial(G)^T$

$\partial(G) = \text{incidence matrix (boundary matrix)}$

“Reduced”: remove rows/columns corresponding to any one vertex.
Example ($K_{3,2}$)

\[ \partial = \begin{pmatrix} 11 & 12 & 21 & 22 & 31 & 32 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 & -1 & 0 \\ 3 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \]

\[ L = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 2 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & -1 & 0 & 3 \end{pmatrix} \]

det($L_r$) = 12, the number of spanning trees of $K_{3,2}$.
Example $(K_{3,2})$

$$
\begin{pmatrix}
2 & 0 & 0 & -1 & -1 \\
0 & 2 & 0 & -1 & -1 \\
0 & 0 & 2 & -1 & -1 \\
-1 & -1 & -1 & 3 & 0 \\
-1 & -1 & -1 & 0 & 3
\end{pmatrix}
\begin{pmatrix}
11 & 12 & 21 & 22 & 31 & 32 \\
1 & -1 & -1 & 0 & 0 & 0 \\
2 & 0 & 0 & -1 & -1 & 0 \\
3 & 0 & 0 & 0 & 0 & -1 \\
1 & 1 & 0 & 1 & 0 & 1 \\
2 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 0 & -1 & -1 \\
0 & 2 & -1 & -1 \\
-1 & -1 & 3 & 0 \\
-1 & -1 & 0 & 3
\end{pmatrix}
$$

$$
\partial =
\begin{bmatrix}
1 & -1 & -1 & 0 & 0 & 0 \\
2 & 0 & 0 & -1 & -1 & 0 \\
3 & 0 & 0 & 0 & 0 & -1 \\
1 & 1 & 0 & 1 & 0 & 1 \\
2 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
\Rightarrow \det(L_r) = 12,
\text{the number of spanning trees of } K_{3,2}.
$$
**Example \((K_{3,2})\)**

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\hline
\text{1} & \text{2} & \text{1} \\
\text{2} & \text{2} & \text{2} \\
\text{3} & \text{2} & \text{2} \\
\end{array}
\]

\[
\partial = \begin{pmatrix}
1 & -1 & -1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & -1 & -1 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & -1 & -1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 \\
2 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
2 & 0 & 0 & -1 & -1 \\
0 & 2 & 0 & -1 & -1 \\
0 & 0 & 2 & -1 & -1 \\
-1 & -1 & -1 & 3 & 0 \\
-1 & -1 & -1 & 0 & 3 \\
\end{pmatrix}
\]

\[
L_r = \begin{pmatrix}
2 & 0 & -1 & -1 \\
0 & 2 & -1 & -1 \\
-1 & -1 & 3 & 0 \\
-1 & -1 & 0 & 3 \\
\end{pmatrix}
\]

\[
\det(L_r) = 12, \text{ the number of spanning trees of } K_{3,2}.
\]
**Weighted Matrix-Tree Theorem**

\[
\sum_{T \in ST(G)} \text{wt } T = | \det \hat{L}_r(G) |,
\]

where \( \hat{L}_r(G) \) is reduced weighted Laplacian.

**Defn 1:** \( \hat{L}(G) = \hat{D}(G) - \hat{A}(G) \)

\[
\hat{D}(G) = \text{diag}(\hat{\deg} v_1, \ldots, \hat{\deg} v_n)
\]
\[
\hat{\deg} v_i = \sum_{v_i v_j \in E} x_i x_j
\]
\( \hat{A}(G) = \text{adjacency matrix} \)

(entry \( x_i x_j \) for edge \( v_i v_j \))

**Defn 2:** \( \hat{L}(G) = \partial(G)B(G)\partial(G)^T \)

\( \partial(G) = \text{incidence matrix} \)
\( B(G) \) diagonal, indexed by edges,

(entry \( \pm x_i x_j \) for edge \( v_i v_j \))
Example \((\mathcal{K}_{3,2})\)

\[
\hat{L}_r = \begin{pmatrix}
2(1 + 2) & 0 & -21 & -22 \\
0 & 3(1 + 2) & -31 & -32 \\
-21 & -31 & 1(1 + 2 + 3) & 0 \\
-22 & -32 & 0 & 2(1 + 2 + 3)
\end{pmatrix}
\]

\[
\det \hat{L}_r = (12312)(1 + 2 + 3)(1 + 2)^2
\]
Proof of $K_{3,2}$ formula

By “identification of factors” (Martin-Reiner, ’03), to show $(1 + 2)^2$ is a factor of the determinant, we just have to show that the nullspace of this matrix is at least 2, when we set $1 + 2 = 0$. 

\[
\text{det} \begin{pmatrix}
2(1 + 2) & 0 & -21 & -22 \\
0 & 3(1 + 2) & -31 & -32 \\
-21 & -31 & 1(1 + 2 + 3) & 0 \\
-22 & -32 & 0 & 2(1 + 2 + 3)
\end{pmatrix}
\]

\[
2312 \text{det} \begin{pmatrix}
1 + 2 & 0 & -1 & -2 \\
0 & 1 + 2 & -1 & -2 \\
-2 & -3 & 1 + 2 + 3 & 0 \\
-2 & -3 & 0 & 1 + 2 + 3
\end{pmatrix}
\]
Finding null vectors

\[
\begin{pmatrix}
1 + 2 & 0 & -1 & -2 \\
0 & 1 + 2 & -1 & -2 \\
-2 & -3 & 1 + 2 + 3 & 0 \\
-2 & -3 & 0 & 1 + 2 + 3
\end{pmatrix}
\]

Since we removed 2 more rows than columns, nullity is at least 2.

Any null vector \((a, b, c)\) of a \(1 \times 3\) matrix gives null vector \((a, b, c, c)\) of a \(4 \times 4\) matrix. (Remember \(1 + 2 = 0\).)

We now have factors \(12(1 + 2)^2\). To get the blue factors, now pick 1 as the vertex to be removed!
Finding null vectors

\[
\begin{pmatrix}
1 + 2 & 0 & -1 & -2 \\
0 & 1 + 2 & -1 & -2 \\
-2 & -3 & 1 + 2 + 3 & 0 \\
-2 & -3 & 0 & 1 + 2 + 3 \\
\end{pmatrix}
\]

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Finding null vectors

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Finding null vectors

\[
\begin{pmatrix}
1 + 2 & 0 & -1 & -2 \\
0 & 1 + 2 & -1 & -2 \\
-2 & -3 & 1 + 2 + 3 & 0 \\
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\end{pmatrix}
\]

Since we removed 2 more rows than columns, nullity is at least 2. Any null vector \((a, b, c)\) of \(1 \times 3\) matrix gives null vector \((a, b, c, c)\) of \(4 \times 4\) matrix. (Remember \(1 + 2 = 0\).) We now have factors \(12(1 + 2)^2\). To get the blue factors, now pick \(1\) as the vertex to be removed!
Simplicial spanning trees of $K_n^d$ [Kalai, '83]

Let $K_n^d$ denote the complete $d$-dimensional simplicial complex on $n$ vertices. $\Upsilon \subseteq K_n^d$ is a simplicial spanning tree of $K_n^d$ when:

0. $\Upsilon_{(d-1)} = K_n^{d-1}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $|\Upsilon| = \binom{n-1}{d}$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.
Counting simplicial spanning trees of $K^d_n$

**Conjecture** [Bolker ’76]

$$
\sum_{\Upsilon \in \text{SST}(K^d_n)} |\tilde{H}^{d-1}(\Upsilon)|^2 = n^{\binom{n-2}{d}}
$$
Counting simplicial spanning trees of $K^d_n$

**Theorem** [Kalai ’83]

$$\tau(K^d_n) = \sum_{\hat{H}_{d-1}(\Upsilon) \in SST(K^d_n)} |\hat{H}_{d-1}(\Upsilon)|^2 = n^{(n-2)}_d$$
Weighted simplicial spanning trees of $K^d_n$

As before,

$$\text{wt } \Upsilon = \prod_{F \in \Upsilon} \text{wt } F = \prod_{F \in \Upsilon} \left( \prod_{v \in F} x_v \right)$$

Example

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$

$$\text{wt } \Upsilon = x_1^5 x_2^4 x_3^3 x_4 x_5^3$$
Weighted simplicial spanning trees of $K^d_n$

As before,

$$\text{wt } \Upsilon = \prod_{F \in \Upsilon} \text{wt } F = \prod_{F \in \Upsilon} \left( \prod_{v \in F} x_v \right)$$

Example

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$

$$\text{wt } \Upsilon = x_1^5 x_2^4 x_3^3 x_4 x_5^3$$

Theorem (Kalai, '83)

$$\hat{\tau}(K^d_n) := \sum_{T \in SST(K^d_n)} |\tilde{H}_{d-1}(\Upsilon)|^2(\text{wt } \Upsilon)$$

$$= (x_1 \cdots x_n)^{(n-2)} (x_1 + \cdots + x_n)^{(n-2)}$$
Proof

Proof uses determinant of reduced Laplacian of $K^d_n$. “Reduced” now means pick one vertex, and then remove rows/columns corresponding to all $(d - 1)$-dimensional faces containing that vertex.

$L = \partial \partial^T$

$\partial : \Delta_d \rightarrow \Delta_{d-1}$ boundary

$\partial^T : \Delta_{d-1} \rightarrow \Delta_d$ coboundary

Weighted version: Multiply column $F$ of $\partial$ by $x_F$
Example $n = 4, d = 2$ (tetrahedron)

\[
\partial^T = \begin{pmatrix}
12 & 13 & 14 & 23 & 24 & 34 \\
123 & -1 & 1 & 0 & -1 & 0 & 0 \\
124 & -1 & 0 & 1 & 0 & -1 & 0 \\
134 & 0 & -1 & 1 & 0 & 0 & -1 \\
234 & 0 & 0 & 0 & -1 & 1 & -1 \\
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
2 & -1 & -1 & 1 & 1 & 1 & 0 \\
-1 & 2 & -1 & -1 & 0 & 1 \\
-1 & -1 & 2 & 0 & -1 & -1 \\
1 & -1 & 0 & 2 & -1 & 1 \\
1 & 0 & -1 & -1 & 2 & -1 \\
0 & 1 & -1 & 1 & -1 & 2 \\
\end{pmatrix}
\]

\[\det L_r = 4\]
Simplicial spanning trees of arbitrary simplicial complexes

Let $\Delta$ be a $d$-dimensional simplicial complex. $\Upsilon \subseteq \Delta$ is a simplicial spanning tree of $\Delta$ when:

0. $\Upsilon_{d-1} = \Delta_{d-1}$ ("spanning");

1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");

2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");

3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.
Simplicial Matrix-Tree Theorem

Theorem (D.-Klivans-Martin, ’09)

\[ \hat{\tau}(\Delta) = \left| \tilde{H}_{d-2}(\Delta; \mathbb{Z}) \right|^2 \frac{\det \hat{L}_\Gamma}{\left| \tilde{H}_{d-2}(\Gamma; \mathbb{Z}) \right|^2}, \]

where

- \( \Gamma \in \text{SST}(\Delta_{(d-1)}) \)
- \( \partial_\Gamma = \) restriction of \( \partial_d \) to faces not in \( \Gamma \)
- reduced Laplacian \( L_\Gamma = \partial_\Gamma \partial^T_\Gamma \)
- Weighted version: Multiply column \( F \) of \( \partial \) by \( x_F \)

Note: The \( |\tilde{H}_{d-2}| \) terms are often trivial.
Example: Octahedron

- Vertices 1, 2, 1, 2, 1, 2.
- Facets 111, 112, 121, 122, 211, 212, 221, 222.
- $\Gamma = 11, 12, 11, 12, 22$ spanning tree of 1-skeleton, so remove (rows and columns corresponding to) those edges from weighted Laplacian.
- $\det \hat{L}_\Gamma = (121212)^3(1 + 2)(1 + 2)(1 + 2)$. 
Complete colorful complexes

Definition (Adin, ’92)
The complete colorful complex $K_{n_1,\ldots,n_r}$ is a simplicial complex with:

- vertex set $V_1 \cup \ldots \cup V_r$ ($V_i$ is set of vertices of color $i$);
- $|V_i| = n_i$;
- faces are all sets of vertices with no repeated colors.

Example
Octahedron is $K_{222}$. 
Theorem (Adin, ’92)

The top-dimensional spanning trees of $K_{n_1,\ldots,n_r}$ are “counted” by

$$\tau(K_{n_1,\ldots,n_r}) = \prod_{i=1}^{r} n_i \prod_{j \neq i} (n_j - 1).$$

Note: Adin also has a more general formula for dimension less than $r - 1$.

Example

$\tau(K_{222}) = 2^1 \times 2^1 \times 2^1$

$\tau(K_{235}) = 2^{2\cdot4} \times 3^{1\cdot4} \times 5^{1\cdot2}$

$\tau(K_{m,n}) = m^{n-1} \times n^{m-1}$
Theorem (Aalipour-D.)

The top-dimensional spanning trees of \( K_{n_1, \ldots, n_r} \) are “counted” by

\[
\tau(K_{n_1, \ldots, n_r}) = \prod_{i=1}^{r} (x_{i,1} + \cdots + x_{i,n_i}) \prod_{j \neq i} (n_j - 1)(x_{i,1} \cdots x_{i,n_i})(\prod_{j \neq i} n_j) - (\prod_{j \neq i} (n_j - 1)).
\]

Example

\[
\hat{\tau}(K_{235}) = (x_1 + x_2)^{2 \cdot 4}(x_1 x_2)^{3 \cdot 5 - 2 \cdot 4}
\times (y_1 + y_2 + y_3)^{1 \cdot 4}(y_1 y_2 y_3)^{2 \cdot 5 - 1 \cdot 4}
\times (z_1 + \cdots + z_5)^{1 \cdot 2}(z_1 \cdots z_5)^{2 \cdot 3 - 1 \cdot 2}
\]
Codimension-1 spanning tree (Adin)

We will use the weighted simplicial matrix-tree theorem. So first we have to find a codimension-1 spanning tree. But it will be a different tree for each color. For each color’s factors, treat that color as “last”.
Codimension-1 spanning tree (Adin)

We will use the weighted simplicial matrix-tree theorem. So first we have to find a codimension-1 spanning tree. But it will be a different tree for each color. For each color’s factors, treat that color as “last”.

$r = 3$ (1-dimensional spanning tree): Start with 1, and attach to every other vertex, except blue vertices. Then use 1 to connect the remaining blue vertices.
Codimension-1 spanning tree (Adin)

We will use the weighted simplicial matrix-tree theorem. So first we have to find a codimension-1 spanning tree. But it will be a different tree for each color. For each color’s factors, treat that color as “last”.

\( r = 3 \) (1-dimensional spanning tree): Start with 1, and attach to every other vertex, except blue vertices. Then use 1 to connect the remaining blue vertices.

\( r = 4 \) (2-dimensional spanning tree): Start with 1, and attach to every edge with no blue vertices. Then use 1, and attach to all edges using a blue non-1 vertex with a non-red vertex. Finally use 1 with edges with a blue non-1 vertex with a red non-1 vertex.
Continuing proof

The rest of the proof is similar to our $K_{3,2}$ computation:

- Reduce by the spanning tree
Continuing proof

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- Reduce by the spanning tree
- Factor out individual variables from the rows
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- Reduce by the spanning tree
- Factor out individual variables from the rows
- Now apply identification of factors:
Continuing proof

The rest of the proof is similar to our $K_{3,2}$ computation:

- Reduce by the spanning tree
- Factor out individual variables from the rows
- Now apply identification of factors:
  - remove the rows containing variables of the last color (number of rows is degree of sum of variables of this color)
Continuing proof

The rest of the proof is similar to our $K_{3,2}$ computation:

- Reduce by the spanning tree
- Factor out individual variables from the rows
- Now apply identification of factors:
  - remove the rows containing variables of the last color (number of rows is degree of sum of variables of this color)
  - remove “duplicate” rows and columns
Continuing proof

The rest of the proof is similar to our $K_{3,2}$ computation:

▶ Reduce by the spanning tree
▶ Factor out individual variables from the rows
▶ Now apply identification of factors:
  ▶ remove the rows containing variables of the last color (number of rows is degree of sum of variables of this color)
  ▶ remove “duplicate” rows and columns
  ▶ null vectors of resulting matrix can be expanded to null vectors of full reduced matrix.
Final thought

Terry Pratchett, *The Colour of Magic*:
“Do you not know that what you belittle by the name tree is but the mere four-dimensional analogue of a whole multidimensional universe which
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Terry Pratchett, *The Colour of Magic*:
“Do you not know that what you belittle by the name tree is but the mere four-dimensional analogue of a whole multidimensional universe which—no, I can see you do not.”

But, now, you do.