Shifted simplicial complexes and algebraic shifting

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Shifted simplicial complexes and algebraic shifting

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OVERVIEW

- shifted complexes
- algebraic shifting
  - symmetric, exterior
  - iterated Betti numbers
    - characterizations
    - decompositions
- Laplacians
  - shifted complexes and matroids
  - recursion

WHY ALGEBRAIC SHIFTING?

It maps a simplicial complex Γ to a shifted simplicial complex $\Delta(\Gamma)$, such that:

- many algebraic, topological invariants preserved; and
- $\Delta(\Gamma)$ is shifted, so
  - $\Delta(\Gamma)$ is combinatorially simpler; and
  - those algebraic and topological invariants are easily computed from the combinatorics of $\Delta(\Gamma)$.

Example (*Björner-Kalai '88*): To characterize possible pairs of f-vectors and (homology) Betti numbers of simplicial complexes, algebraically shift, preserving f and Betti's. Betti's are now easy to read: facets that don't contain vertex 1.

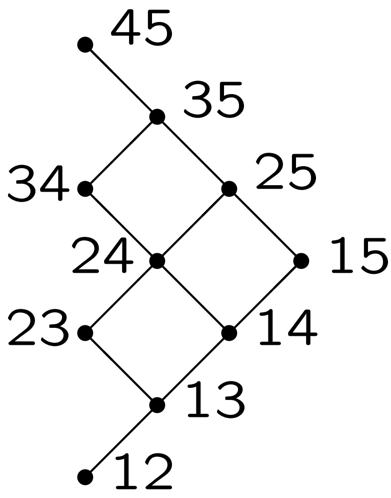
SHIFTED COMPLEXES

Defn: A simplicial complex Γ on ordered vertex set $[n] = \{1, \ldots, n\}$ is **shifted** if

$$F \in \Gamma,\ v' < v \Rightarrow (F - v) \dot{\cup} v' \in \Gamma.$$

Examples:

123	134	234	17
124	135	235	46
125	136	236	
126	145		



Equivalently, Γ is an order ideal in the componentwise partial order on ordered subsets of $[n]$.

NEAR-CONES (Björner-Kalai)

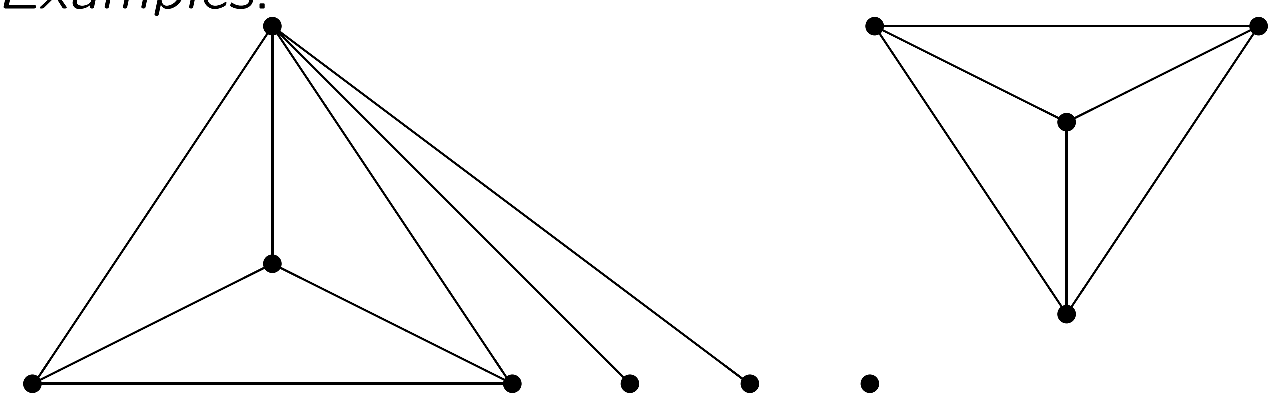
$\Delta' \dot{\cup} B$ any simplicial complex
B some facets (boundary of B in Δ')
near-cone $\Delta = (v * \Delta') \dot{\cup} B$, v apex
(Alternatively: $v * (\Delta' \dot{\cup} B) -$ "$v * B$")

Homology Betti numbers easy to compute:

$$\tilde{\beta}_i(\Delta) := \dim \tilde{H}_i(\Delta) = \# B_i$$

(facets of each dimension, not containing vertex 1).

Examples:



Betti numbers $(1, 1, 0)$ and $(0, 0, 0)$.

SHIFTED IS ITERATED NEAR-CONE

$$\Delta = 1 * \overbrace{((2 * \underbrace{((3 * \Delta''') \dot\cup B_2)}_{\Delta''} \dot\cup B_1)}^{\Delta'} \dot\cup B_0$$

Example:

$$\begin{aligned}
\Delta''' &= \{\emptyset\} \\
\Delta'' &= 3 * \Delta''' \dot\cup \{4, 5, 6\} \\
\Delta' &= 2 * \Delta'' \dot\cup \{34, 35, 36, 45, 7\} \\
\Delta &= 1 * \Delta' \dot\cup \{234, 235, 236, 46\}
\end{aligned}$$

123	134	234
124	135	235
125	136	236
126	145	46
	17	

SHIFTED $\Rightarrow$ SHELLABLE

Shelling is a way to build a simplicial complex one facet at a time, so that as each facet F is added, a unique new minimal face $R(F)$ is added. Therefore $\Delta = \dot{\cup}_s [R(F_s), F_s]$. For a shellable complex (Björner-Wachs '96),

$$h_{i,r}(\Delta) = \#\{\text{facets } F \colon |R(F)| = r,\ |F| = i\}.$$

A shifted complex has a canonical shelling, by arranging facets in lexicographic order. In this shelling, $R(F) = F - [\text{init}(F)]$. Thus, for a shifted complex,

$$h_{i,r}(\Delta) = \#\{\text{facets } F \colon \text{init}(F) = i - r,\ |F| = i\}.$$

Example:

123	134	234	17
124	135	235	46
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126	145		

ALGEBRAIC SHIFTING (Kalai)

Map simplicial complex Γ to new simplicial complex $\Delta^e(\Gamma)$.

Exterior version Stanley-Reisner face-ring:

$$x^F := \wedge_{i \in F} x_i$$
$$I_\Gamma := \langle x^F : F \notin \Gamma \rangle$$
$$\Lambda[\Gamma] := \Lambda[x_1, \ldots, x_n]/I_\Gamma$$

$$y_j := \alpha_{j1}x_1 + \alpha_{j2}x_2 + \cdots + \alpha_{jn}x_n,$$

where α_{ij}'s "generic" (for instance, added algebraically indpt. variables). Define $\Delta^{(e)}$ by:

$$F \in \Delta^e(\Gamma) \Leftrightarrow y^F \notin \text{span}\{y^G : G <_{\text{lex}} F\} \text{ in } \Lambda[\Gamma]$$

Equivalently,

$$I_{\Delta^e(\Gamma)} = \text{Gin}_{\text{revlex}}(I_\Gamma) = \text{in}_{\text{revlex}}(gI_\Gamma),$$

where g is "generic" in $GL(n)$.

(Gin is generic initial ideal, and in is initial ideal, from Gröbner basis theory.)

SYMMETRIC ALGEBRAIC SHIFTING

(Kalai '91, but also recent papers by Herzog, et. al.)

Use ordinary (commutative) Stanley-Reisner face-ring, and then, almost as before,

$$I_{\Delta^s(\Gamma)} = \phi \mathrm{Gin}_{\mathrm{revlex}}(I_\Gamma),$$

where ϕ is the "squarefree" operator, e.g.,

$$\phi(y_4 y_6^3 y_7) = \phi(y_4 y_6 y_6 y_6 y_7)$$
$$= y_{4-4} y_{6-3} y_{6-2} y_{6-1} y_{7-0} = y_0 y_3 y_4 y_5 y_7$$

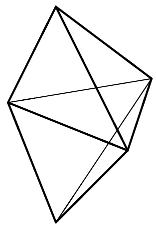
This is of interest for graded ideals other than I_Γ (i.e., not squarefree), without applying ϕ. In other words $\mathrm{Gin}_{\mathrm{revlex}}$ is an interesting operator on arbitrary graded ideals. "Strongly stable" ideal is commutative algebra version of shifted complex, and $\mathrm{Gin}(I)$ is strongly stable.

“AXIOMS” OF ALGEBRAIC SHIFTING

1. $\Delta(\Gamma)$ is a shifted simplicial complex;
2. $\Gamma' \subseteq \Gamma \Rightarrow \Delta(\Gamma') \subseteq \Delta(\Gamma)$;
3. $f(\Gamma) = f(\Delta(\Gamma))$, $\tilde{\beta}(\Gamma) = \tilde{\beta}(\Delta(\Gamma))$;
4. $\mathrm{Cone}(\Delta(\Gamma)) = \Delta(\mathrm{Cone}(\Gamma))$;
5. Γ shifted $\Rightarrow \Delta(\Gamma) = \Gamma$;
6. Γ Cohen-Macaulay iff $\Delta(\Gamma)$ CM (note that a shifted complex is CM iff it is pure).

(Note: 5 follows from the previous properties!)

Example: Alg. shift triangular bipyramid.



ITERATED BETTI NUMBERS

Algebraic version of deconing of shifted complexes. Exterior (D-Rose '00; earlier version Kalai) or symmetric (Babson-Novik-Thomas).

Take homology (exterior) or local cohomology (symmetric) of exterior or symmetric face-ring using the first generic variable, then algebraically "peel off" that variable. Repeat until there's nothing left. The iterated Betti numbers are the dimensions of homology or local cohomology at each step.

Equivalently, read the iterated Betti numbers of $\Delta(\Gamma)$:

$$\begin{aligned} b_{i,r}(\Gamma) &= h_{i,r}(\Delta(\Gamma)) \\ &= \#\{\text{facets } F\colon \text{init}(F) = i-r,\ |F| = i\}. \end{aligned}$$

$$F = \{1, 2, \ldots, i-r; k_1, \ldots, k_r\}$$

ALGEBRAIC BETTI NUMBERS

$\beta_{i,j}(\Gamma)$, invariants of minimal free resolution of $k[\Gamma]$ over S:

$$0 \to F_t \to \cdots \to F_1 \to S^1 \to k[\Gamma] \to 0$$
$$F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$$

It was long known that algebraic Betti numbers, in addition to being important in and of themselves, determine many interesting invariants of $k[\Gamma]$ (equivalently, Γ):

- Krull dimension $(1 + \dim \Gamma)$;
- Hilbert series (f-vector);
- homological dimension;
- regularity;
- depth; and
- topological Betti numbers of Γ.

EXTREMAL NUMBERS

Extremal algebraic Betti numbers (*Bayer-Charalambous-Popescu* '99): $\beta_{i,j} \neq 0$, can't make i or $j-i$ bigger, without making $\beta = 0$. (Southeast corner of Macaulay diagram.)

Easy to show these are enough to find invariants of previous slide, except for the Hilbert series (f-vector).

Extremal symmetric iterated Betti numbers (*BNT*): $b^s_{i,r}(\Gamma) = h_{i,r}(\Delta^s(\Gamma)) \neq 0$, can't make i smaller, or r bigger, without making $b = 0$. (Recall this counts facets $F = [i-r] \dot\cup R$ in $\Delta^s(\Gamma)$, with $|F| = i, |R| = r$).

Thm (*BNT*): Extremal symmetric iterated Betti numbers determine the extremal algebraic Betti numbers.

CHARACTERIZING INVARIANTS

Characterize those **many** invariants of **all** simplicial complexes.

reduced to

Characterize **extremal** h-**numbers** (or, **"init"** data) of all **shifted** complexes.

extremal h (or, "init") of $\Delta^s(\Gamma)$

$\rightarrow$ extremal symm. iterated Bettis of Γ

$\rightarrow$ extremal Betti numbers of Γ

$\rightarrow$ various invariants of Γ

Extremal: $h_{i,r} \neq 0$, can't make i smaller, or r bigger, without making $h = 0$. (Recall this counts facets $F = [i-r] \dot\cup R$, with $|F| = i, |R| = r$).

DECOMPOSITIONS: COHEN-MACAULAY COMPLEXES

Conj (*Stanley* '79, *Garsia* '80): If Δ is Cohen-Macaulay, then it may be partitioned

$$\Delta = \dot{\cup}_s [R_s, F_s]$$

($[R_s, F_s]$ denotes a Boolean interval) where each F_s is a facet of Δ, and so

$$h_j(\Delta) = \#\{s\colon |R_s| = j\}.$$

For a pure shellable complex,

$$h_j(\Delta) = h_{d,j}(\Delta) = \#\{\text{facets } F\colon |R(F)| = j\}.$$

DECOMPOSITIONS: ALL COMPLEXES

Conj (*Kalai* '93): If

$$\Delta^e(\Gamma) = \dot{\cup}_s [F_s, G_s]$$

is a "nice" partition of $\Delta(\Gamma)$, then there is a partition

$$\Gamma = \dot{\cup}_s [A_s, B_s]$$

where

$$\dim A_s = \dim F_s, \ \dim B_s = \dim G_s.$$

"Nice" includes the canonical shelling of a shifted complex, where $R(F) = F - [\mathrm{init}(F)]$, so

Conj (*D-Zhang* '01): Any simplicial complex Γ can be partitioned

$$\Gamma = \dot{\cup}_s [A_s, B_s]$$

where

$$\#\{s \colon |A_s| = j, \ |B_s| = i\} = h_{i,j}(\Delta^e(\Gamma)) = b^e_{i,j}(\Gamma).$$

Prop DZ $\Rightarrow$ Garsia-Stanley

BOOLEAN TREES

Thm (*D-Zhang*): Any simplicial complex Γ can be partitioned $\Gamma = \dot\cup_s T_s$, where each T_s is a Boolean tree, and

$$\#\{s\colon |\min(T_s)| = j,\ |\max(T_s)| = i\} = b^e_{i,j}(K).$$

Cor: If Δ is Cohen-Macaulay, then it may be partitioned $\Delta = \dot\cup T_s$, where each T_s is a Boolean tree whose maximal element is a facet of Δ, and

$$\#\{s\colon |\min(T_s)| = j\} = h_j(\Delta).$$

Question: How much of this can we repeat for **symmetric** iterated Betti's? Especially since Cohen-Macaulay is defined in symmetric (commutative) algebra. Note BNT conjecture: $b^s_{i,j} \leq b^e_{i,j}$.

LAPLACIANS

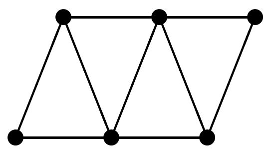
$C_i = C\Gamma_i$, the i-dimensional $\mathbb{R}$-chains of Γ ($\mathbb{R}$-linear combinations of i-dim'l faces of Γ)

$\partial = \partial_i \colon C_i \to C_{i-1}$ usual signed boundary
$\delta_{i-1} = \partial_i^* \colon C_{i-1} \to C_i$ coboundary.

$$C_{i+1} \underset{\partial^*}{\overset{\partial}{\rightleftharpoons}} C_i \underset{\partial^*}{\overset{\partial}{\rightleftharpoons}} C_{i-1}$$

Defn: i-dimensional **Laplacian** of Γ:

$$L_i(\Gamma) = \partial_{i+1}\partial_{i+1}^* + \partial_i^*\partial_i \colon C_i \to C_i$$



Easy observations about s, eigenvalues

$\mathrm{s}(L_i) = \mathrm{s}(\partial_{i+1}\partial_{i+1}^*) \cup\ \mathrm{s}(\partial_i^*\partial_i)$

$\mathrm{s}(\partial_i^*\partial_i) = \mathrm{s}(\partial_i\partial_i^*)$, except for 0's, and number of 0 eigenvalues is ith Betti number.
So we may as well just consider $s_i'' = \mathrm{s}(\partial_i^*\partial_i)$, which only depends on C_i.

DEGREE SEQUENCES

Defn: d_i is the i-**dimensional degree sequence** $(d_i)_j = \#$ i-faces containing vertex j.

Example:

123	134	234
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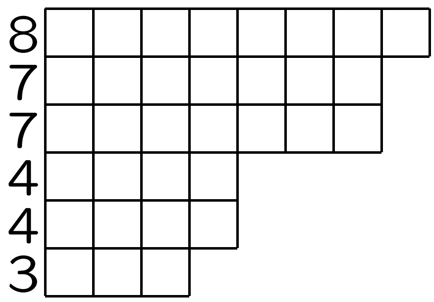


$$s'' = (6, 6, 6, 5, 3, 3, 3, 1, 0, 0, 0)$$

Thm (*D-Reiner* '02): If a simplicial complex is shifted, then

$$s''_i = (d_i)^T.$$

Also integral for

- **matroids** (*Kook-Reiner-Stanton* '00)
- chessboard complexes (*Friedman-Hanlon* '98)
- matching complexes of K_n (*Dong-Wachs* '02).

MATROIDS AND SHIFTED COMPLEXES

Both: collection of subsets of ground set $E = \{1, \ldots, n\}$.

Matroid bases $\mathcal{B}$: $\forall B \in \mathcal{B},\ \forall b \in B$
$\forall B' \in \mathcal{B},\ \exists b' \in B'$ such that

$$(B - b) \cup b' \in \mathcal{B}.$$

Shifted family K: $\forall F \in K,\ \forall v \in F$
$\forall v' < v,$ if $v' \notin F,$ then

$$(F - v) \cup v' \in K.$$

Independence complex IN(M) of matroid M is simplicial complex of subsets of bases (these are the independent sets).

Simplicial complex Δ is shifted if it is shifted in every dimension. Note that taking all subsets of the members of a shifted family gives a shifted simplicial complex.

SPECTRAL RECURSION FOR MATROIDS...

$$S_M(t,q) := \sum_i t^i \sum_{\lambda \in \mathbf{s}(L_{i-1}(\mathsf{IN}(M)))} q^\lambda$$

Tutte-Grothendieck invariants have a deletion-contraction recursion:

$$\phi(M) = \phi(M-e) + \phi(M/e)$$

$$\mathcal{B}(M-e) = \{B \in \mathcal{B} \colon e \notin B\} \qquad (r = r(M))$$
$$\mathcal{B}(M/e) = \{B - e \colon B \in \mathcal{B},\ e \in B\} \quad (r = (M) - 1)$$

Thm (*Kook*): $S_M = qS_{M-e} + qtS_{M/e}$
$+(1-q)$(error term).

Conj (*Kook-Reiner*): error term $= S_{(M-e,M/e)}$, where $(M-e, M/e) = (\mathsf{IN}(M-e), \mathsf{IN}(M/e))$ is the "relative complex" of IN$(M-e)$ with all the faces from IN(M/e) removed.

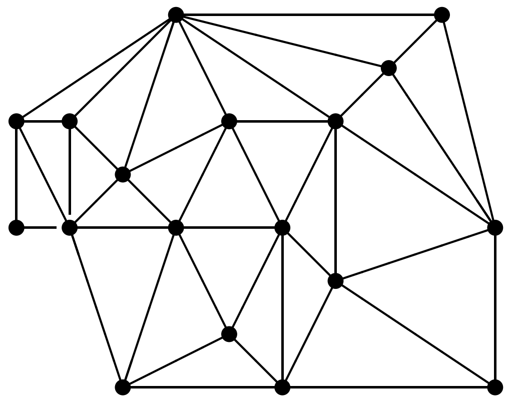
Thm: This is true, *i.e.*,

$$S_M = qS_{M-e} + qtS_{M/e} + (1-q)S_{(M-e,M/e)}.$$

...AND FOR SHIFTED COMPLEXES

Generalize deletion and contraction to arbitrary simplicial complex Γ.

$$\Gamma - e = \{F \in \Gamma \colon e \notin F\}$$
$$\Gamma/e = \{F - e \colon F \in \Gamma,\ e \in F\} \qquad = \mathrm{lk}_\Gamma\, e$$



$$S_\Gamma(t,q) := \sum_i t^i \sum_{\lambda \in \mathbf{s}(L_{i-1}(\Gamma))} q^\lambda$$

Thm: Spectral recursion holds for shifted complexes Δ:

$$S_\Delta = qS_{\Delta-e} + qtS_{\Delta/e} + (1-q)S_{(\Delta-e,\Delta/e)}.$$

COMMON GENERALIZATION?

What else has integral Laplacian spectrum, and satisfies the spectral recursion? Call such complexes integral, and spectral, respectively.

Defn: **join** $\Gamma_1 * \Gamma_2 = \{F_1 \dot\cup F_2 \colon F_i \in \Gamma_i\}$
Thm: If Γ_1 and Γ_2 are integral (resp., spectral), then so is $\Gamma_1 * \Gamma_2$.

disjoint union
Thm: If Γ_1 and Γ_2 are integral (resp., spectral), then so is $\Gamma_1 \dot\cup \Gamma_2$.

Defn: i-**skeleton** $\Gamma^{(i)} := \{F \in \Gamma \colon \dim F \leq i\}$.
Thm: If Γ is integral (resp., spectral), then so is $\Gamma^{(i)}$, for every i.

DUALS

Recall matroid dual: $\mathcal{B}^* = \{E - B \colon B \in \mathcal{B}\}$.

Defns: **dual** $\Gamma^* := \{E - F \colon F \in \Gamma\}$ (simplicial complex $\leftrightarrow$ order filter)

complement $\Gamma^c = \{F \subseteq E \colon F \not\in \Gamma\}$ (simplicial complex $\leftrightarrow$ order filter)

Alexander dual $\Gamma^\vee := \Gamma^{*c} = \Gamma^{c*}$ (simplicial complex $\leftrightarrow$ simplicial complex)

Thm: Γ integral (resp., spectral) iff Γ^* integral (resp., spectral) iff Γ^c integral (resp., spectral).

(modify spectral recursion, slightly, for order filters:

$$S_{\Gamma^*} = qS_{\Gamma^*-e} + qtS_{\Gamma^*/e} + (1-q)tS_{(\Gamma^*/e, \Gamma^*-e)})$$

Cor: Γ integral (resp., spectral) iff $\Gamma^\vee$ is.

Rmk: Γ is shifted iff $\Gamma^\vee$ is