A non-partitionable Cohen-Macaulay simplicial complex

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Stanley: "I am glad that this problem has finally been put to rest, though I would have preferred a proof rather than a counterexample. Perhaps you can withdraw your paper from the arXiv and come up with a proof instead."

Simplicial complexes

Definition (Simplicial complex)

Let V be set of vertices. Then Δ is a simplicial complex on V if:

- $\Delta \subseteq 2^V$; and
- if $\sigma \subseteq \tau \in \Delta$ implies $\sigma \in \Delta$.

Higher-dimensional analogue of graph.

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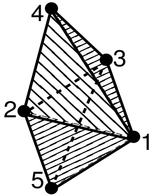
Definition (f-vector)

$f_i = f_i(\Delta) =$ number of i-dimensional faces of Δ. The f-vector of $(d-1)$-dimensional Δ is

$$f(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$$

Example



124, 125, 134, 135, 234, 235;
12, 13, 14, 15, 23, 24, 25, 34, 35;
1, 2, 3, 4, 5;
$\emptyset$

$$f(\Delta) = (1, 5, 9, 6)$$

Counting faces of spheres

Definition (Sphere)

Simplicial complex whose realization is a triangulation of a sphere.

Conjecture (Upper Bound)

Explicit upper bound on f_i *of a sphere with given dimension and number of vertices.*

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This was proved by Stanley in 1975. Some of the key ingredients:

- face-ring (algebraic object derived from the simplicial complex) [Stanley, Hochster]
- face-ring of sphere is Cohen-Macaulay [Reisner]

Cohen-Macaulay simplicial complexes

CM *rings* of great interest in commutative algebra (depth = dimension). Here is a more topological/combinatorial definition.

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$\mathrm{lk}_\Delta \sigma = \{\tau \in \Delta \colon \tau \cap \sigma = \emptyset,\ \tau \cup \sigma \in \Delta\}$, what Δ looks like near σ.

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Theorem (Reisner '76)

Face-ring of Δ is Cohen-Macaulay if, for all $\sigma \in \Delta$,

$$\tilde{H}_i(\mathrm{lk}_\Delta\, \sigma) = 0 \quad \textit{for } i < \dim \mathrm{lk}_\Delta\, \sigma.$$

We take this as our definition of CM simplicial complex.

Cohen-Macaulayness is topological

Recall our definition:

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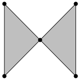
Munkres ('84) showed that CM is a topological condition. That is, it only depends on (the homeomorphism class of) the realization of Δ. In particular, spheres and balls are CM.

Example

 is not CM

h-vector

The conditions for the UBC most easily stated in terms of h-vector.

Definition (h-vector)

Let $\dim \Delta = d - 1$.

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{k=0}^{d} h_k t^{d-k}$$

The h-vector of Δ is $h(\Delta) = (h_0, h_1, \ldots, h_d)$. Coefficients not always non-negative!

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Equivalently,

$$\sum_{i=0}^{d} f_{i-1} t^{d-i} = \sum_{k=0}^{d} h_k (t+1)^{d-k}$$

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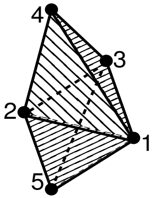
Example



$f(\Delta) = (1, 5, 9, 6)$, and

$$1t^3 + 5t^2 + 9t + 6 = {\color{red}1}(t+1)^3 + {\color{blue}2}(t+1)^2 + {\color{green}2}(t+1)^1 + 1$$

so $h(\Delta) = ({\color{red}1}, {\color{blue}2}, {\color{green}2}, 1)$.

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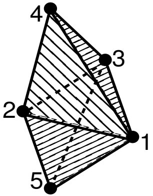


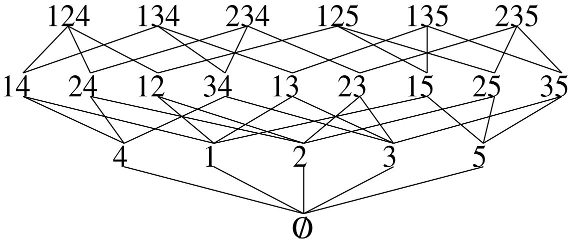
$f(\Delta) = (1, 5, 9, 6)$, and

$$1t^3 + 5t^2 + 9t + 6 = 1(t+1)^3 + 2(t+1)^2 + 2(t+1)^1 + 1$$

so $h(\Delta) = (1, 2, 2, 1)$.
Note that in this case, $h \geq 0$. This is a consequence of the algebraic defn of CM. But how could we see this combinatorially?

Partitionability

$$1t^3 + 5t^2 + 9t + 6 = \textcolor{red}{1}(t+1)^3 + \textcolor{blue}{2}(t+1)^2 + \textcolor{green}{2}(t+1)^1 + 1$$



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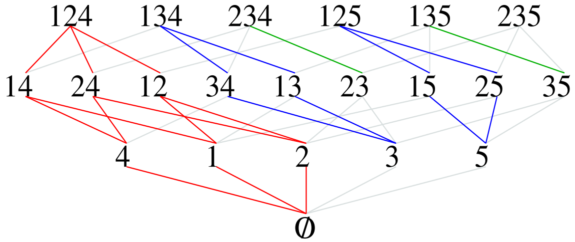

Definition (Partitionable)

When a simplicial complex can be partitioned like this, into Boolean intervals whose tops are facets, we say the complex is partitionable.

Shellability

Most CM complexes in combinatorics are shellable:

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A simplicial complex is shellable if it can be built one facet at a time, so that there is always a unique new minimal face being added.

A shelling is a particular kind of partitioning.

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Proposition

If Δ is shellable, then h_k counts number of intervals whose bottom (the unique new minimal face) is dimension $k-1$.

Example

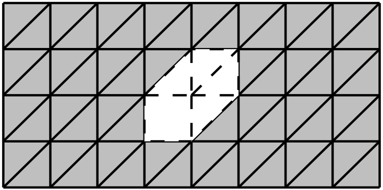
In our previous example, minimal new faces were: $\emptyset$, vertex, edge, vertex, edge, triangle.

We were trying to prove the conjecture

Idea of our "proof":

- Remove all the faces containing a given vertex (this will be the first part of the partitioning).

Example



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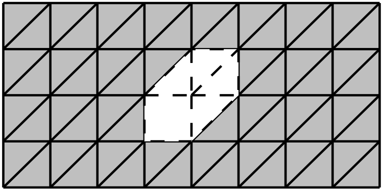

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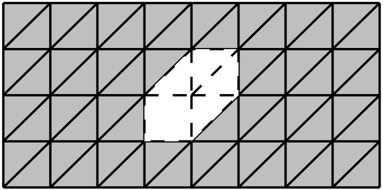

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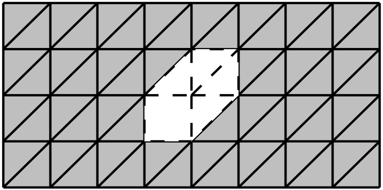
The problem is we would have to prove the conjecture for relative CM complexes.

Example



Relative simplicial complexes

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Φ is a relative simplicial complex on V if:

- $\Phi \subseteq 2^V$; and
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We can write any relative complex Φ as $\Phi = (\Delta, \Gamma)$, for some pair of simplicial complexes $\Gamma \subseteq \Delta$.

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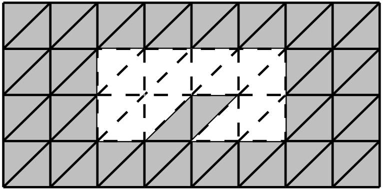

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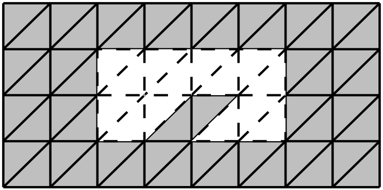

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This generalizes easily:

Theorem (Stanley '87)

Face-ring of $\Phi = (\Delta, \Gamma)$ is relative Cohen-Macaulay if, for all $\sigma \in \Delta$,

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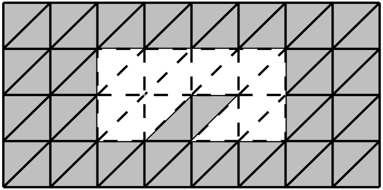

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- We wanted to find a non-trivial example of something Cohen-Macaulay and partitionable, so we could see how this idea of relative complexes would work.
- How hard is it to take that second step of the partitioning, which is the first step for the relative complex?
- Idea: non-trivial = not shellable; CM = ball (and if it's not partitionable, we're done). So we are looking for non-shellable balls.

M.E. Rudin's non-shellable ball ('58)

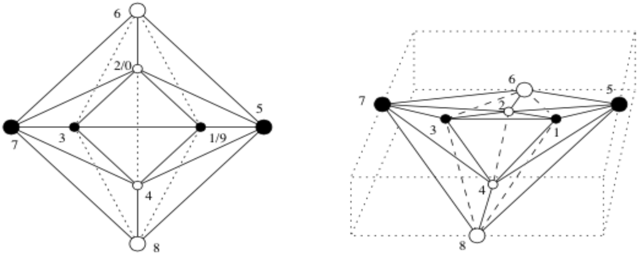
First we tried M.E. Rudin's non-shellable 3-ball:

- 3-dimensional (built out of tetrahedra);
- 14 vertices;
- 41 tetrahedra;
- Can be realized as triangulation of tetrahedron with all vertices on boundary.

Did not help.

Ziegler's non-shellable ball ('98)

Non-shellable 3-ball with 10 vertices and 21 tetrahedra



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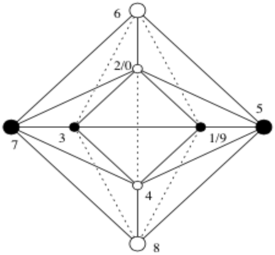
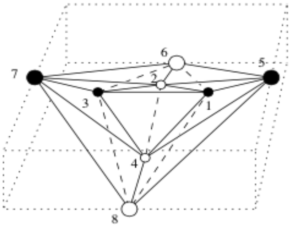


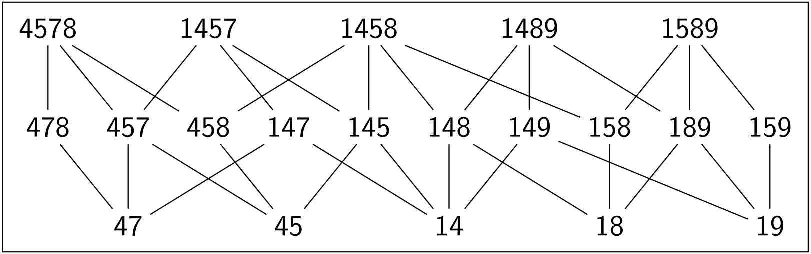


Just because it is partitionable does not mean you can start partitioning in any order.
So we started to partition until we could not go any further (without backtracking). This part uses the computer!

First pass with Ziegler

We found a relative complex $Q_5 = (X_5, A_5)$

- X_5 has 6 vertices, 5 facets



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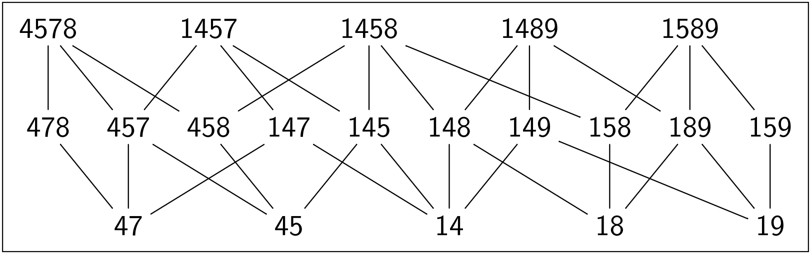

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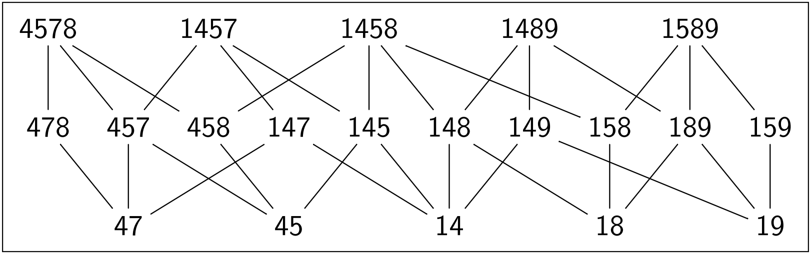

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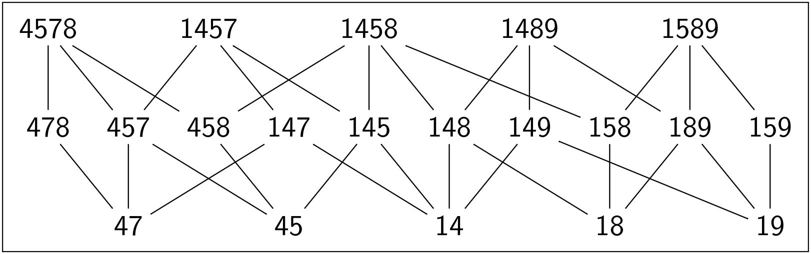

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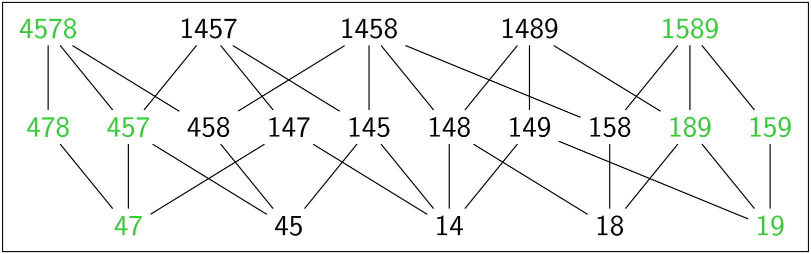

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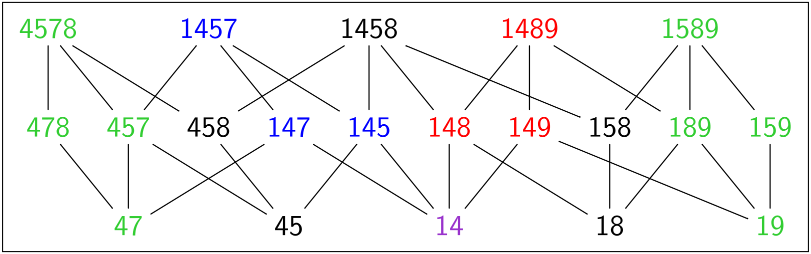

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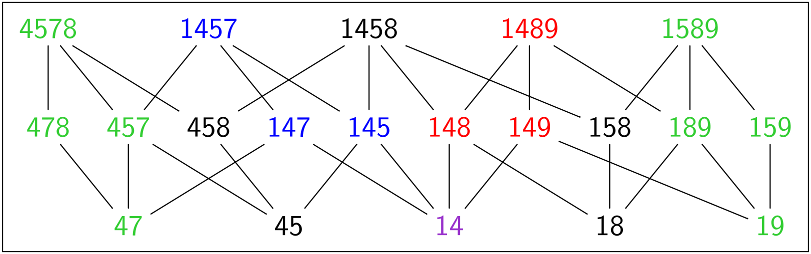

If only we could build a non-relative complex out of this.

Gluing

Proposition

If X and (X, A) are CM and $\dim A = \dim X - 1$, then gluing together two copies of X along A gives a CM (non-relative) complex.

If we glue together two copies of X along A, is it partitionable?

Gluing

Proposition

If X and (X, A) are CM and $\dim A = \dim X - 1$, then gluing together two copies of X along A gives a CM (non-relative) complex.

If we glue together two copies of X along A, is it partitionable? Maybe: some parts of A can help partition one copy of X, other parts of A can help partition the other copy of X.

Pigeonhole principle

Recall our example (X, A) is:

- relative Cohen-Macaulay
- not partitionable

Remark

If we glue together many copies of X along A, at least one copy will be missing all of A!

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Remark

But the resulting complex is not actually a simplicial complex because of repeats.

Pigeonhole principle

Need our example (X, A) to be:

- relative Cohen-Macaulay
- not partitionable
- A vertex-induced (minimal faces of (X, A) are vertices)

Remark

If we glue together many copies of X along A, at least one copy will be missing all of A! How many is enough? More than the number of all faces in A. Then the result will not be partitionable.

Remark

But the resulting complex is not actually a simplicial complex because of repeats. To avoid this problem, we need to make sure that A is vertex-induced. This means every face in X among vertices in A must be in A as well. (Minimal faces of (X, A) are vertices.)

Eureka!

By computer search, we found that if

- Z is Ziegler's 3-ball, and
- $B = Z$ restricted to all vertices except 1,5,9 (B has 7 facets),

then $Q = (Z, B)$ satisfies all our criteria!

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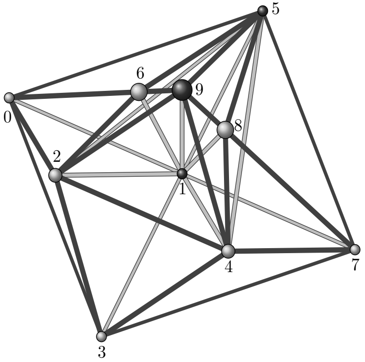
Also $Q = (X, A)$, where X has 14 facets, and A is 5 triangles:



1249	1269
1569	1589
1489	1458
1457	4578
1256	0125
0256	0123
1234	1347

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- Since A has 24 faces total (including the empty face), we know gluing together 25 copies of X along their common copy of A, the resulting (non-relative) complex C_{25} is CM, not partitionable.

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- In fact, computer search showed that gluing together only 3 copies of X will do it. Resulting complex C_3 has f-vector $(1, 16, 71, 98, 42)$.
- Later we found short proof by hand to show that C_3 works.

Stanley depth (a brief summary)

Definition (Stanley)

If I is a monomial ideal in a polynomial ring S, then the Stanley depth $\operatorname{sdepth} S/I$ is a purely combinatorial analogue of depth, defined in terms of certain vector space decompositions of S/I.

Conjecture (Stanley '82)

For all monomial ideals I, $\operatorname{sdepth} S/I \geq \operatorname{depth} S/I$.

Theorem (Herzog, Jahan, Yassemi '08)

If I is the Stanley-Reisner ideal (related to the face ring) of a Cohen-Macaulay complex Δ, then the inequality $\operatorname{sdepth} S/I \geq \operatorname{depth} S/I$ *is equivalent to the partitionability of Δ.*

Corollary

Our counterexample disproves this conjecture as well.

Constructibility

Definition

A d-dimensional simplicial complex Δ is constructible if:

- it is a simplex; or
- $\Delta = \Delta_1 \cup \Delta_2$, where $\Delta_1, \Delta_2, \Delta_1 \cap \Delta_2$ are constructible of dimensions $d, d, d-1$, respectively.

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Question (Hachimori '00)

Are constructible complexes partitionable?

Corollary

Our counterexample is constructible, so the answer to this question is no.

Open question: Smaller counterexample?

Open questions:

Question

Is there a smaller 3-dimensional counterexample to the partitionability conjecture?

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Is there a smaller 3-dimensional counterexample to the partitionability conjecture?

Question

Is the partitionability conjecture true in 2 dimensions?

Save the conjecture: Strengthen the hypothesis

More open questions (based on what our counterexample is not):
Note that our counterexample is not a ball (3 balls sharing common 2-dimensional faces), but all balls are CM.

Question

Are simplicial balls partitionable?

Balanced CM complexes

Garsia's version of conjecture was for CM posets, which lead to balanced complexes.

Definition (Balanced)

A simplicial complex is balanced if vertices can be colored so that every facet has one vertex of each color.

Question

Are balanced Cohen-Macaulay complexes partitionable?

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Definition (Balanced)

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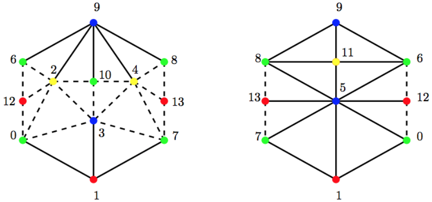
Question

Are balanced Cohen-Macaulay complexes partitionable?

Theorem (Juhnke-Kubitzke, Venturello '19)

No.



Save the conjecture: Weaken the conclusion

Question

What does the h-vector of a CM complex count?

Save the conjecture: Weaken the conclusion

Question

What *does* *the h-vector of a CM complex count?*

One possible answer (D.-Zhang '01) replaces Boolean intervals with "Boolean trees". But maybe there are other answers.