Variations on a G-theme: The G-Shi arrangement, and its relation to G-parking functions

Art Duval[1] Caroline Klivans[2] Jeremy Martin[3]

[1]University of Texas at El Paso

[2]University of Chicago

[3]University of Kansas

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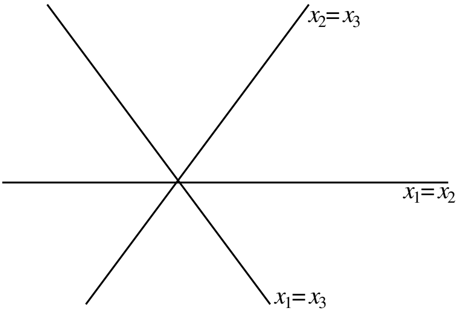
Hyperplane arrangements

In $\mathbb{R}^n$, a finite collection, or **arrangement**, $\mathcal{A}$ of hyperplanes partition the complement of $\mathcal{A}$ into a finite number of regions. For many naturally defined arrangements, the number of regions is interesting.

Braid arrangement

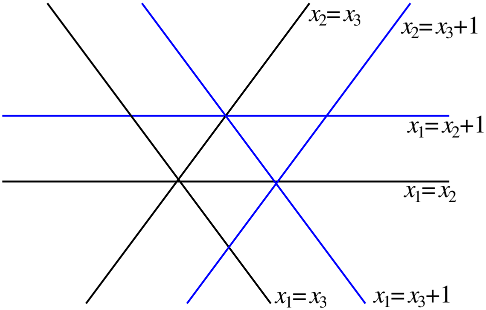
$$\{x_i = x_j : 1 \leq i < j \leq n\}$$

has $n!$ regions, one for every total ordering of the variables $x_1, \ldots, x_n$.



Shi arrangement

$$\mathcal{S}_n := \{x_i = x_j, x_i = x_j + 1 : 1 \leq i < j \leq n\}$$



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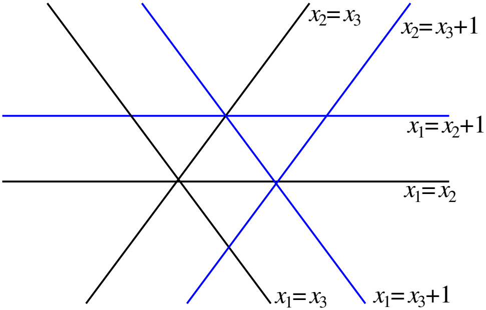


Theorem (Shi, '86)

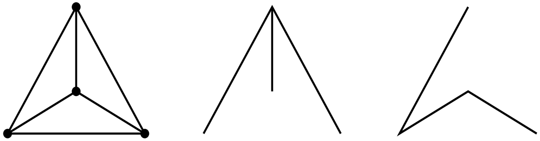
The Shi arrangement $\mathcal{S}_n$ has $(n+1)^{n-1}$ regions.

Spanning trees

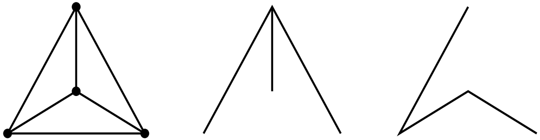
Definition

A set T of edges of a graph $G = (V, E)$ is a **spanning tree** of G if (the endpoints of) T contains all of V and [equivalently, any two of the following three]:

- T has no cycles
- T is connected
- $|T| = |V| - 1$



Cayley's theorem



For K_4, there are

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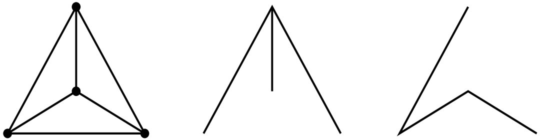

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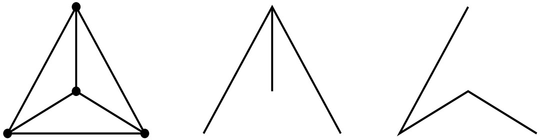

For K_4, there are

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- 12 path spanning trees

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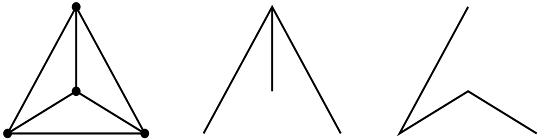

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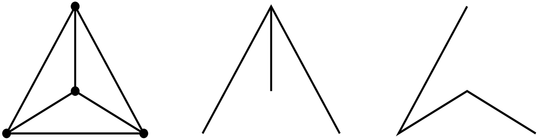

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Theorem (Cayley)

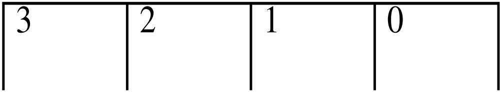
The complete graph K_{n+1} on $n+1$ vertices has $(n+1)^{n-1}$ spanning trees.

Parking functions

Definition

- parking spots $0, \ldots, n-1$

Example



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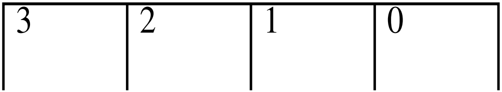

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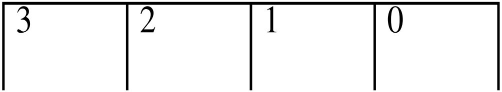
1120



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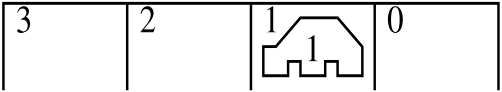

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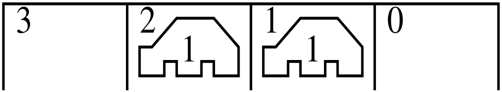

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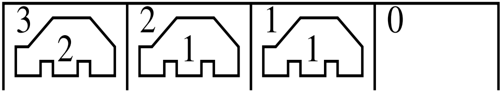

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If such a function f allows all the cars to park, it is a **parking function**. [Note that indexing is sometimes different.]

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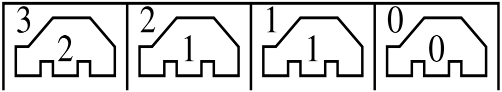
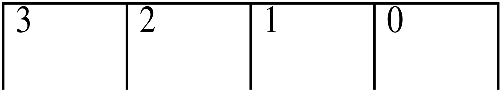


Which functions are parking functions?



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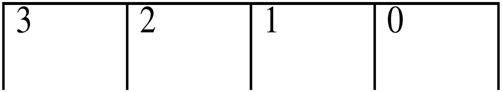

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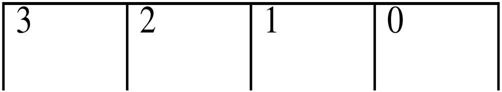

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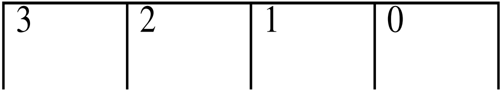

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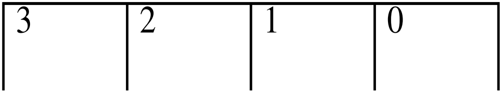

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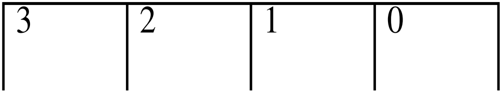

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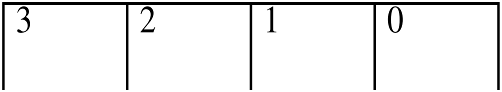

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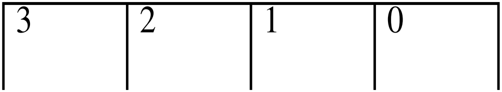

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This is sufficient, too (making values less only makes it easier to park).

How many are there?

n=2: 00, 01, 10

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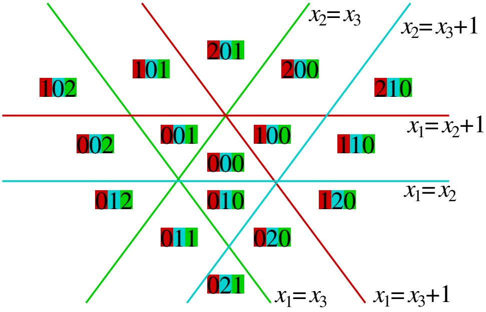
Theorem (Pyke, '59; Konheim and Weis, '66)

There are $(n+1)^{n-1}$ *parking functions.*

Pak-Stanley labelling

Pak and Stanley found a labelling of the regions of the Shi arrangement so that each region gets a different label,



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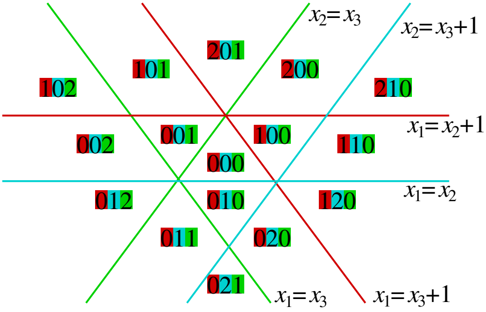

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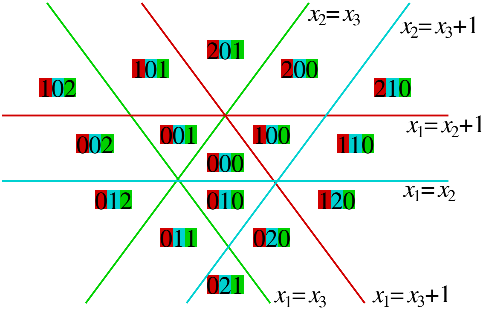

Athanasiadis and Linusson have alternate (easier) bijection between parking functions and Shi regions.

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How much of this generalizes to arbitrary graphs? What does that even mean?

Restating parking function definition

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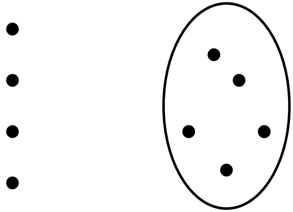
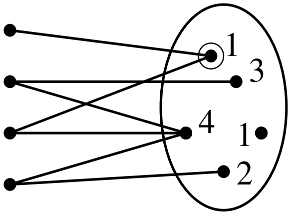


G-parking functions

Definition

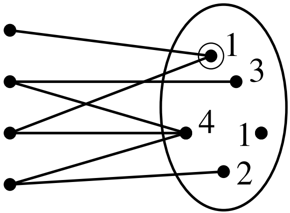
Given a graph $G = (V, E)$, , a function $f : V \rightarrow \mathbb{Z}^{\geq 0}$ is a **parking function** if, in any set $U \subseteq V$ of vertices, there is at least one vertex v such that $f(v)$ is at most the $\bar{U}$-degree of v, the number of neighbors of v outside of U.



G-parking functions

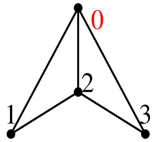
Definition

Given a graph $G = (V, E)$, with root q, a function $f : V \backslash q \to \mathbb{Z}^{\geq 0}$ is a **parking function** if, in any set $U \subseteq V \setminus q$ of vertices, there is at least one vertex v such that $f(v)$ is at most the $\bar{U}$-degree of v, the number of neighbors of v outside of U.



Note that if $G = K_{n+1}$ we get classical parking functions on n cars.

Example



0	0	0
0	0	1
0	1	0
1	0	0
0	1	1
1	0	1
1	1	0
0	2	0

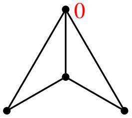
Critical group

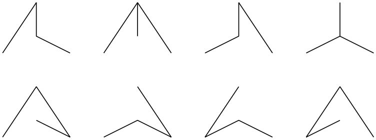
Where does this come from?
In the chip-firing game, every vertex (except the root vertex) of a graph has a number of chips. A vertex may "fire", sending one chip to every neighbor, as long as it has enough chips to do so. If we consider the set of all arrangements of chips, but declare two arrangements to be equivalent if you can get from one to the other by a series of firings, we get a quotient group, called the critical group.

Theorem (Dhar, '90)

The set of G-parking functions form a particularly nice set of representatives of the critical group. The order of this group is the number of spanning trees of G.

Example





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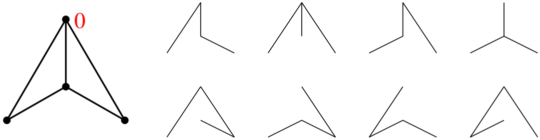

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Our motivation was to generalize this result to higher dimensions (parking functions, critical group, spanning trees).

Question

How will we generalize the Shi arrangement to an arbitrary graph $G = (V, E)$?

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Recall, in complete graph case, the labels of regions of the Shi arrangement $\mathcal{S}_n$ are given by the parking functions on n cars. The number of elements in either set equals the number of spanning trees of the complete graph K_{n+1}.

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For an arbitrary graph, we now have the number of spanning trees of G equals the number of G-parking functions. What about the Shi arrangement?

Graphical arrangement

Start with braid arrangement, but include only hyperplanes corresponding to edges in graph:

$$\{x_i = x_j \colon i < j; \{i, j\} \in E\}$$

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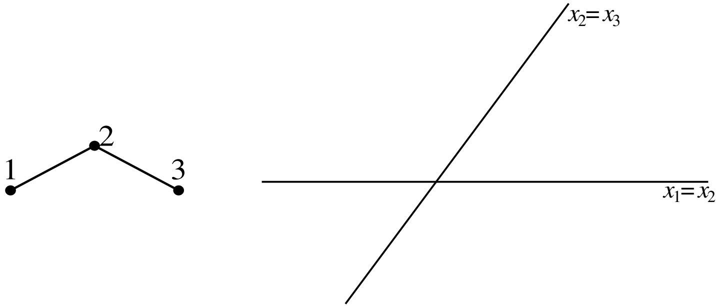
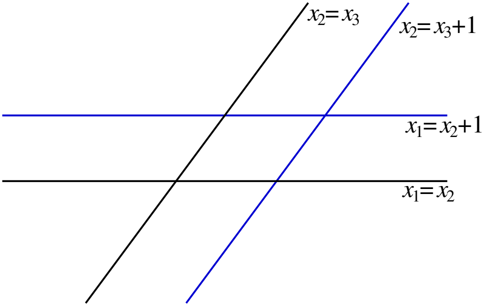


G-Shi arrangement

If we combine the ideas of the graphical arrangement and the Shi arrangement, we get

$$\{x_i = x_j, x_i = x_j + 1 \colon i < j; \{i,j\} \in E\}$$



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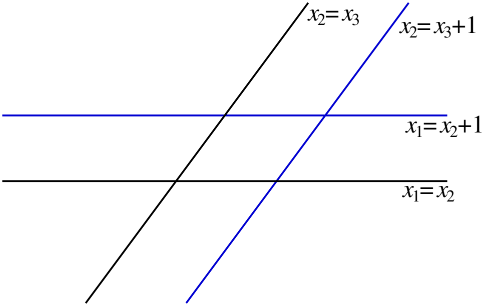
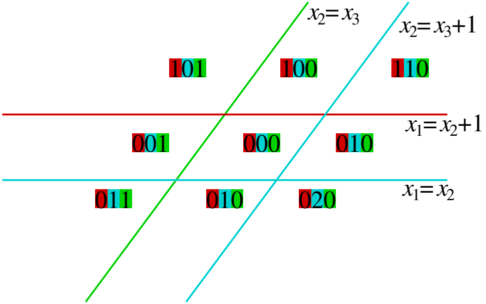


But this has 9 regions, and there are only 8 spanning trees and 8 parking functions.

Labels

What if we put in the analogs of the Pak-Stanley labels?



Conjecture

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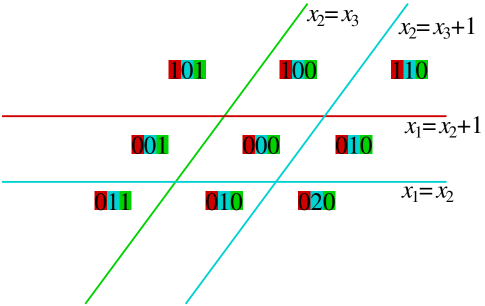
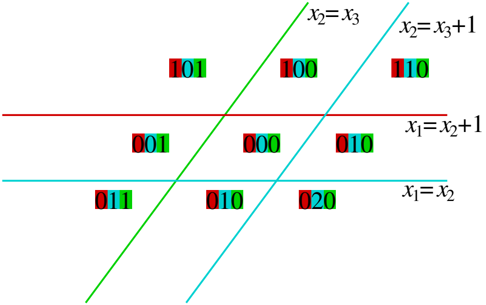


Conjecture

There is a bijection between the $(0 * G)$-parking functions and the set of different labels of the G-Shi arrangement.

Maximal labels in G-Shi

What are the maximal labels (maximum total weight) of the G-Shi arrangement?



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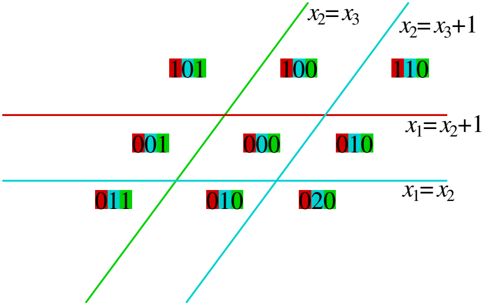

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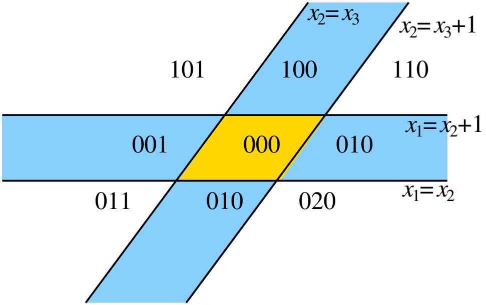

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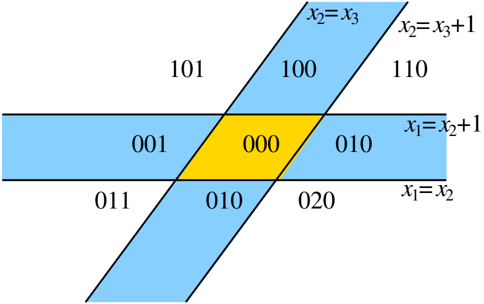
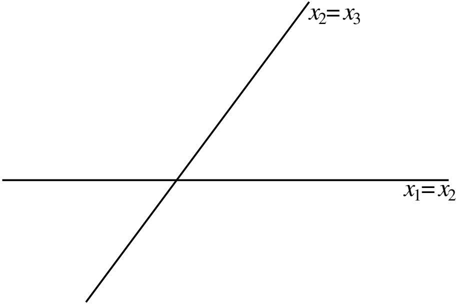


For every pair of parallel hyperplanes (corresponding to edge in graph), you have to be on one side or the other:

Graphical arrangement



For every pair of parallel hyperplanes (corresponding to edge in graph), you have to be on one side or the other: Graphical arrangement!

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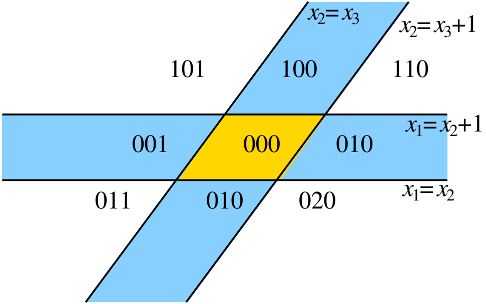

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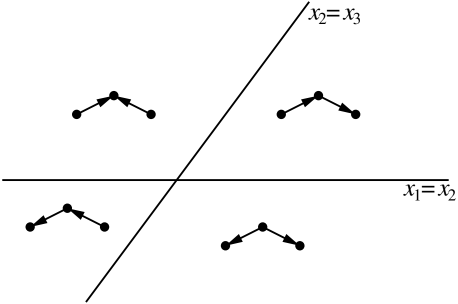
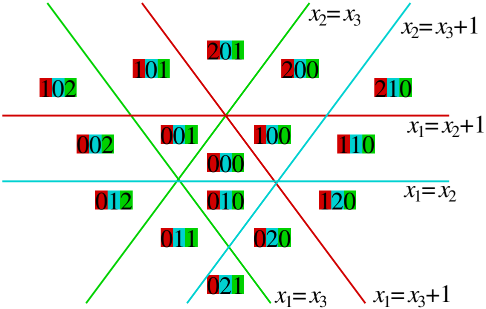
Weight goes up by one for every hyperplane crossed, so total weight is number of edges of G.

Acyclic orientations

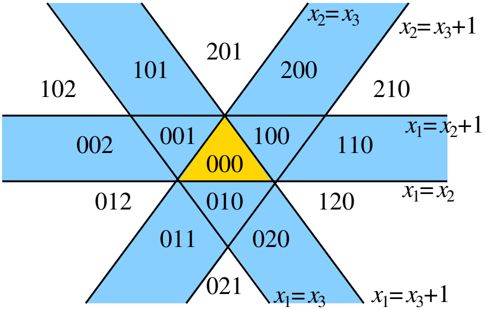


Regions of graphical arrangement correspond to acyclic orientations on graph (just like regions of braid arrangement correspond to permutations, which correspond to acyclic orientations of the complete graph).
So there is a natural bijection between maximal labels of the G-Shi arrangement and acyclic orientations of G.

Example: K_n again



Example: K_n again



Maximal G-parking functions

Theorem (Benson, Chakrabarty, Tetali, '10)

Maximal $0 * G$-parking functions also have weight equal to the number of edges of G, and correspond to acyclic orientations of G.

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Observation (Easy)

If f is a G-parking function, and $g(v) \leq f(v)$ for all v, then g is also a G-parking function

Proof.

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Consequence: If we could only show that labels also satisfy the easy observation, we'd be done.

Half the bijection

We can use this to easily show that every label g has a corresponding parking function:
There exists some maximal label f such that $g(v) \leq f(v)$ for all v ($g = f$ is possible). Since f is maximal, it corresponds to an acyclic orientation O. By BCT, we know O corresponds to a maximal parking function, so f is a maximal parking function. By the easy observation, g is also a parking function.

What about the other half?

We still need to show either [equivalently]:

- Every parking function is a label
- Labels satisfy the easy observation