Counting weighted simplicial spanning trees of shifted complexes

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Counting weighted simplicial spanning trees of shifted complexes

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**MATRIX TREE THEOREM**

\[ \sum_{T \in ST(G)} \text{wt } T = | \det \hat{L}_r(G) |, \]

where

\[ \text{wt } T = \prod_{e \in T} \text{wt } e = \prod_{e \in T} \left( \prod_{v \in e} x_v \right), \]

"Reduced": remove rows/columns corresponding to any one vertex

\( \hat{L} = \partial \partial^T \), (weighted) Laplacian
\( \partial \), weighted boundary (incidence) matrix; the \((e, v)\) entry is \( \pm \sqrt{\text{wt } e} \)

Example:

![Graph](image)

\[ \partial^T = \begin{pmatrix}
-\sqrt{12} & \sqrt{12} & 0 & 0 \\
-\sqrt{13} & 0 & \sqrt{13} & 0 \\
-\sqrt{14} & 0 & 0 & \sqrt{14} \\
0 & -\sqrt{23} & \sqrt{23} & 0 \\
0 & -\sqrt{24} & 0 & \sqrt{24}
\end{pmatrix} \]
EXAMPLE

\[ \hat{L} = \begin{pmatrix} 1(2 + 3 + 4) & -12 & -13 & -14 \\ -12 & 2(1 + 3 + 4) & -23 & -24 \\ -13 & -23 & 3(1 + 2) & 0 \\ -14 & -24 & 0 & 4(1 + 2) \end{pmatrix} \]

\[ \hat{L}_r = \begin{pmatrix} 2(1 + 3 + 4) & -23 & -24 \\ -23 & 3(1 + 2) & 0 \\ -24 & 0 & 4(1 + 2) \end{pmatrix} \]

\[ \det \hat{L}_r = (1234)(1 + 2)(1 + 2 + 3 + 4) \]
SIMPLICIAL SPANNING TREES of COMPLETE SIMPLICIAL COMPLEX
(Kalai ‘83)

Defn: Simplicial spanning tree: (Assume dim $k$; $n$ vertices.) Set $T$, of $k$-dimensional faces, containing all $(k-1)$-dimensional faces and:

1. $|T| = \binom{n-1}{k}$

2. $\tilde{H}_k(T) = 0$

3. $\tilde{H}_{k-1}(T)$ is finite group

Note: Any two conditions imply the third.

How many are there?
Bolker (‘76): should be $n\binom{n-2}{k}$, but isn’t

Kalai (‘83): proves

$$\sum_{T \in SST} |\tilde{H}_{k-1}(T)|^2 = n\binom{n-2}{k}$$
WEIGHTING

$$\text{wt } T = \prod_{F \in T} \text{wt } F = \prod_{F \in T} (\prod_{v \in F} x_v)$$

Thm (Kalai ’83):

$$\sum_{T \in SST} |\tilde{H}_{k-1}(T)|^2(\text{wt } T)$$

$$= (x_1 \cdots x_n)^{(n-2)}(x_1 + \cdots + x_n)^{(n-k)}$$

Adin (’92) did something similar for complete r-partite complexes.
KALAI’S THEOREM

Proof (unweighted; weighted is similar):

\[ n^{(n-2)}_k = \det L_r(K_n^k) = \det \partial_r(K_n^k)\partial_r((K_n^k)^T) \]
\[ = \sum_T (\det \partial_r(T))^2 \]
\[ = \sum_T |\tilde{H}_{k-1}(T)|^2 \]

by Binet-Cauchy.

“Reduced” now means pick one vertex, and then remove rows/columns corresponding to all \((k-1)\)-dimensional faces containing that vertex.

\[ L = \partial \partial^T \]

\(\partial: \Delta_k \rightarrow \Delta_{k-1} \) boundary
\(\partial^T: \Delta_{k-1} \rightarrow \Delta_k \) coboundary
SIMPLICIAL MATRIX TREE THEOREM

Defn: Simplicial spanning tree of $\Delta$: (Assume $\dim \Delta = k$.) $k$-dimensional complex $T$ containing all $(k-1)$-dimensional faces of $\Delta$ ($T^{(k-1)} = \Delta^{(k-1)}$) and:

1. $f_k(T) = f_k(\Delta) - \beta_k(\Delta) + \beta_{k-1}(\Delta)$
2. $\tilde{H}_k(T) = 0$
3. $\tilde{H}_{k-1}(T)$ is finite group

Note: Any two conditions imply the third.

Thm (DKM):

$$\sum_{T \in SST(\Delta)} |\tilde{H}_{k-1}(T)|^2 wt T = \frac{|\tilde{H}_{k-2}(\Delta)|^2}{|\tilde{H}_{k-2}(\Delta_U)|^2} \det \hat{L}_r.$$

$U$ = set of facets of $(k-1)$-spanning tree of $\Delta$
$\hat{L}_r$ is $L$ reduced by all of $U$
$\Delta_U = U \cup \Delta^{(k-2)}$
SHIFTED SIMPLICIAL COMPLEXES

Defn: \( V = 1, \ldots, n \)
\( F \in \Delta, i \notin F, j \in F, i < j \Rightarrow F \cup i - j \in \Delta \)
(equivalently, the \( k \)-faces form an initial ideal in the componentwise partial order).

Ex: bipyramid = \{ 123, 124, 125, 134, 135, 234, 235 \} (and subfaces)

\[ \Delta = (1 \ast \text{lk}_\Delta 1) \cup B_\Delta \]

\( B_\Delta = \{ F \in \Delta : 1 \notin F, F \cup 1 \notin \Delta \} \)
\( \text{lk}_\Delta 1 = \{ F - 1 : 1 \in F, F \in \Delta \} \), shifted
\( \text{del}_\Delta 1 = \{ G : 1 \notin G, G \in \Delta \} \), shifted
\( \beta_i(\Delta) = f_i(B_\Delta) \)
\( \text{del}_\Delta 1 = \text{lk}_\Delta 1 \cup B_\Delta \)

Example (bipyramid, continued):
\( B_\Delta = \{ 234, 235 \} \)
\( \text{lk}_\Delta 1 = \{ 23, 24, 25, 34, 25; 2, 3, 4, 5; \emptyset \} \)
\( \text{del}_\Delta 1 = \text{lk}_\Delta 1 \cup B_\Delta \).
EIGENVALUES

Thm (D-Reiner ‘02): Non-0 eigenvalues of top-dim’l Laplacian of shifted complex given by $d^T$ where $d$ is degree sequence, $d_i = |\{\text{facets } F : i \in F\}|$.

Example (bipyramid):
$\{123, 124, 125, 134, 135, 234, 235\}$

Example (del1): 234, 235
COUNTING TREES OF SHIFTED COMPLEXES

Pick $U$ to be ridges $((k-1)$-dimensional faces) containing 1, which is acyclic, and contains $(k-2)$-faces of $\Delta$, and so it is (the facets of) a simplicial spanning tree.

Also, $\hat{H}_{k-2}(\Delta_U) = 0$, and if $\Delta$ is pure and shifted, then $\hat{H}_{k-2}(\Delta) = 0$, so we just have to compute $\det \hat{L}_r$.

Ex: bipyramid. Set of all ridges is all possible edges, except 45. $U = \{12, 13, 14, 15\}$, so $\hat{L}_r$ is indexed by $\{23, 24, 25, 34, 35\}$.

$$\hat{L}_r =$$

$$\begin{array}{cccccc}
23(1+4+5) & -234 & -235 & 234 & 235 \\
-234 & 24(1+3) & 0 & -234 & 0 \\
-235 & 0 & 25(1+3) & 0 & -235 \\
234 & 234 & 0 & (1+2)34 & 0 \\
235 & 0 & -235 & 0 & (1+2)35
\end{array}$$
SIMPLIFICATIONS

\[ \det \hat{L}_r = (23)(24)(25)(34)(35) \det M \]

\[
\det M = \begin{vmatrix}
1 + 4 + 5 & -\sqrt{34} & -\sqrt{35} & \sqrt{24} & \sqrt{25} \\
-\sqrt{34} & 1 + 3 & 0 & -\sqrt{23} & 0 \\
-\sqrt{35} & 0 & 1 + 3 & 0 & -\sqrt{23} \\
\sqrt{24} & -\sqrt{23} & 0 & 1 + 2 & 0 \\
\sqrt{25} & 0 & -\sqrt{23} & 0 & 1 + 2 \\
\end{vmatrix} = |1I + N|
\]

\[
N = \begin{pmatrix}
4 + 5 & -\sqrt{34} & -\sqrt{35} & \sqrt{24} & \sqrt{25} \\
-\sqrt{34} & 3 & 0 & -\sqrt{23} & 0 \\
-\sqrt{35} & 0 & 3 & 0 & -\sqrt{23} \\
\sqrt{24} & -\sqrt{23} & 0 & 2 & 0 \\
\sqrt{25} & 0 & -\sqrt{23} & 0 & 2 \\
\end{pmatrix}
\]

Remarkably, \( N \) is a weighted Laplacian of \( \det_{\Delta 1} \); \( N = \partial \partial^T \) with

\[
\partial^T = \begin{pmatrix}
\sqrt{4} & -\sqrt{3} & 0 & \sqrt{2} & 0 \\
0 & -\sqrt{3} & 0 & 0 & \sqrt{2} \\
\end{pmatrix}
\]

The \((F, G)\) entry of this matrix is \( \pm \sqrt{F - G} \).
FURTHER SIMPLIFICATIONS

This makes the eigenvalues of this $N$ be 0, 0, 0, 2 + 3, 2 + 3 + 4 + 5, and
\[ \det M = (1)^3(1 + 2 + 3)(1 + 2 + 3 + 4 + 5). \]
Finally it makes the weighted tree enumerator of the bipyramid

\[ (23)(24)(25)(34)(35) \times (1)^3(1 + 2 + 3)(1 + 2 + 3 + 4 + 5). \]

More generally,

\[
\left( \prod_{F \in R} x_F \right)^{|R|} \prod_{j=1} 1 + (d^T)_j \sum_{i=1} x_i,
\]

where $R =$ facets of $\text{lk}_{\Delta} 1$ and $d$ is the degree sequence of $\text{del}_{\Delta} 1$.  

\[ \begin{array}{c}
4 & 2 \\
3 & \end{array} \quad \begin{array}{c}
2 & 3 \\
4 & 5 \\
5 & \end{array} \]