Counting weighted simplicial spanning trees of shifted complexes

AMS Southeastern Section Meeting

Davidson College

Mar., '07

Counting weighted simplicial spanning trees of shifted complexes

Art Duval
University of Texas at El Paso

Caroline Klivans
University of Chicago

Jeremy Martin
University of Kansas

MATRIX TREE THEOREM

$$\sum_{T \in ST(G)} \operatorname{wt} T = |\det \hat{L}_r(G)|,$$

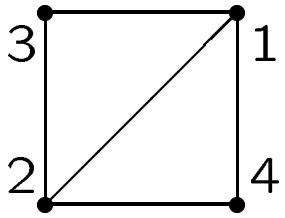
where

$$\operatorname{wt} T = \prod_{e \in T} \operatorname{wt} e = \prod_{e \in T} (\prod_{v \in e} x_v),$$

"Reduced": remove rows/columns corresponding to any one vertex

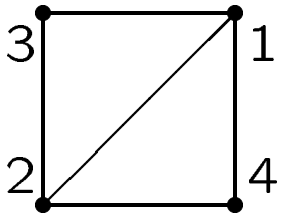
$\hat{L} = \partial \partial^T$, (weighted) Laplacian
∂, weighted boundary (incidence) matrix;
the (e, v) entry is $\pm\sqrt{\operatorname{wt} e}$

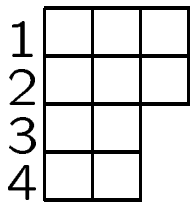
Example:



$$\partial^T = \begin{pmatrix} -\sqrt{12} & \sqrt{12} & 0 & 0 \\ -\sqrt{13} & 0 & \sqrt{13} & 0 \\ -\sqrt{14} & 0 & 0 & \sqrt{14} \\ 0 & -\sqrt{23} & \sqrt{23} & 0 \\ 0 & -\sqrt{24} & 0 & \sqrt{24} \end{pmatrix}$$

EXAMPLE

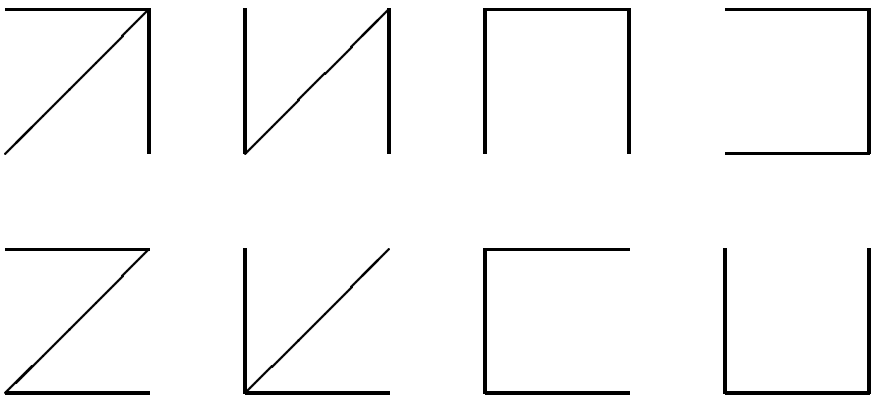




$$\widehat{L} = \begin{pmatrix} 1(2+3+4) & -12 & -13 & -14 \\ -12 & 2(1+3+4) & -23 & -24 \\ -13 & -23 & 3(1+2) & 0 \\ -14 & -24 & 0 & 4(1+2) \end{pmatrix}$$

$$\widehat{L}_r = \begin{pmatrix} 2(1+3+4) & -23 & -24 \\ -23 & 3(1+2) & 0 \\ -24 & 0 & 4(1+2) \end{pmatrix}$$

$$\det \widehat{L}_r = (1234)(1+2)(1+2+3+4)$$



SIMPLICIAL SPANNING TREES of COMPLETE SIMPLICIAL COMPLEX (Kalai '83)

Defn: Simplicial spanning tree: (Assume dim k; n vertices.) Set T, of k-dimensional faces, containing all $(k-1)$-dimensional faces and:

1. $|T| = \binom{n-1}{k}$
2. $\tilde{H}_k(T) = 0$
3. $\tilde{H}_{k-1}(T)$ is finite group

Note: Any two conditions imply the third.

How many are there?
Bolker ('76): should be $n^{\binom{n-2}{k}}$, but isn't

Kalai ('83): proves

$$\sum_{T \in SST} |\tilde{H}_{k-1}(T)|^2 = n^{\binom{n-2}{k}}$$

WEIGHTING

$$\text{wt}\, T = \prod_{F \in T} \text{wt}\, F = \prod_{F \in T} \left(\prod_{v \in F} x_v \right)$$

Thm (Kalai '83):

$$\sum_{T \in SST} |\tilde{H}_{k-1}(T)|^2 (\text{wt}\, T)$$
$$= (x_1 \cdots x_n)^{\binom{n-2}{k-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{k}}$$

Adin ('92) did something similar for complete r-partite complexes.

KALAI'S THEOREM

Proof (unweighted; weighted is similar):

$$\begin{aligned} n^{\binom{n-2}{k}} &= \det L_r(K_n^k) = \det \partial_r(K_n^k)\partial_r(K_n^k)^T \\ &= \sum_T (\det \partial_r(T))^2 \\ &= \sum_T |\tilde{H}_{k-1}(T)|^2 \end{aligned}$$

by Binet-Cauchy.

"Reduced" now means pick one vertex, and then remove rows/columns corresponding to all $(k-1)$-dimensional faces containing that vertex.

$L = \partial\partial^T$

∂: $\Delta_k \to \Delta_{k-1}$ boundary
∂^T: $\Delta_{k-1} \to \Delta_k$ coboundary

SIMPLICIAL MATRIX TREE THEOREM

Defn: Simplicial spanning tree of Δ: (Assume $\dim \Delta = k$.) k-dimensional complex T containing all $(k-1)$-dimensional faces of Δ ($T^{(k-1)} = \Delta^{(k-1)}$) and:

1. $f_k(T) = f_k(\Delta) - \beta_k(\Delta) + \beta_{k-1}(\Delta)$
2. $\tilde{H}_k(T) = 0$
3. $\tilde{H}_{k-1}(T)$ is finite group

Note: Any two conditions imply the third.

Thm (DKM):

$$\sum_{T \in SST(\Delta)} |\tilde{H}_{k-1}(T)|^2 \mathrm{wt}\, T = \frac{|\tilde{H}_{k-2}(\Delta)|^2}{|\tilde{H}_{k-2}(\Delta_U)|^2} \det \hat{L}_r.$$

U =set of facets of $(k-1)$-spanning tree of Δ
$\hat{L}_r$ is L reduced by all of U
$\Delta_U = U \cup \Delta^{(k-2)}$

SHIFTED SIMPLICIAL COMPLEXES

Defn: $V = 1, \ldots, n$
$F \in \Delta, i \notin F, j \in F, i < j \Rightarrow F \cup i - j \in \Delta$
(equivalently, the k-faces form an initial ideal in the componentwise partial order).

Ex: bipyramid = { 123, 124, 125, 134, 135, 234, 235} (and subfaces)

$$\Delta = (1 * \mathrm{lk}_\Delta 1) \dot\cup B_\Delta$$

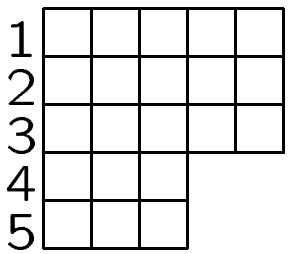
$B_\Delta = \{F \in \Delta \colon 1 \notin F, F \dot\cup 1 \notin \Delta\}$
$\mathrm{lk}_\Delta 1 = \{F - 1 \colon 1 \in F, F \in \Delta\}$, shifted
$\mathrm{del}_\Delta 1 = \{G \colon 1 \notin G, G \in \Delta\}$, shifted
$\beta_i(\Delta) = f_i(B_\Delta)$
$\mathrm{del}_\Delta 1 = \mathrm{lk}_\Delta 1 \dot\cup B_\Delta$

Example (bipyramid, continued):
$B_\Delta = \{234, 235\}$
$\mathrm{lk}_\Delta 1 = \{23, 24, 25, 34, 25; 2, 3, 4, 5; \emptyset\}$
$\mathrm{del}_\Delta 1 = \mathrm{lk}_\Delta 1 \dot\cup B_\Delta$.

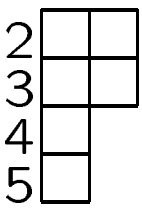
EIGENVALUES

Thm (D-Reiner '02): Non-0 eigenvalues of top-dim'l Laplacian of shifted complex given by d^T where d is degree sequence, $d_i = |\{\text{facets } F : i \in F\}|$.

Example (bipyramid):
$\{123, 124, 125, 134, 135, 234, 235\}$



Example (del1): $234, 235$



COUNTING TREES OF SHIFTED COMPLEXES

Pick U to be ridges ($(k-1)$-dimensional faces) containing 1, which is acyclic, and contains $(k-2)$-faces of Δ, and so it is (the facets of) a simplicial spanning tree.

Also, $\tilde{H}_{k-2}(\Delta_U) = 0$, and if Δ is pure and shifted, then $\tilde{H}_{k-2}(\Delta) = 0$, so we just have to compute $\det \hat{L}_r$.

Ex: bipyramid. Set of all ridges is all possible edges, except 45. $U = \{12, 13, 14, 15\}$, so $\hat{L}_r$ is indexed by $\{23, 24, 25, 34, 35\}$.

$\hat{L}_r =$

$23(1+4+5)$	-234	-235	234	235
-234	$24(1+3)$	0	-234	0
-235	0	$25(1+3)$	0	-235
234	234	0	$(1+2)34$	0
235	0	-235	0	$(1+2)35$

SIMPLIFICATIONS

$$\det \widehat{L}_r = (23)(24)(25)(34)(35) \det M$$

$$\det M = \begin{vmatrix} 1+4+5 & -\sqrt{34} & -\sqrt{35} & \sqrt{24} & \sqrt{25} \\ -\sqrt{34} & 1+3 & 0 & -\sqrt{23} & 0 \\ -\sqrt{35} & 0 & 1+3 & 0 & -\sqrt{23} \\ \sqrt{24} & -\sqrt{23} & 0 & 1+2 & 0 \\ \sqrt{25} & 0 & -\sqrt{23} & 0 & 1+2 \end{vmatrix}$$

$$= \left| 1I + N \right|$$

$$N = \begin{pmatrix} 4+5 & -\sqrt{34} & -\sqrt{35} & \sqrt{24} & \sqrt{25} \\ -\sqrt{34} & 3 & 0 & -\sqrt{23} & 0 \\ -\sqrt{35} & 0 & 3 & 0 & -\sqrt{23} \\ \sqrt{24} & -\sqrt{23} & 0 & 2 & 0 \\ \sqrt{25} & 0 & -\sqrt{23} & 0 & 2 \end{pmatrix}$$

Remarkably, N is a weighted Laplacian of $\mathrm{del}_\Delta 1$; $N = \partial\partial^T$ with

$$\partial^T = \begin{pmatrix} \sqrt{4} & -\sqrt{3} & 0 & \sqrt{2} & 0 \\ \sqrt{5} & 0 & -\sqrt{3} & 0 & \sqrt{2} \end{pmatrix}$$

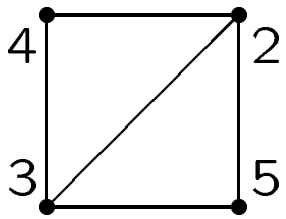
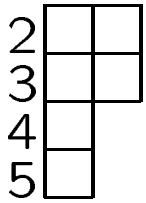
The (F, G) entry of this matrix is $\pm\sqrt{F-G}$.

FURTHER SIMPLIFICATIONS

This makes the eigenvalues of this N be
0, 0, 0, $2+3$, $2+3+4+5$, and
$\det M = (1)^3(1+2+3)(1+2+3+4+5)$.
Finally it makes the weighted tree enumerator of the bipyramid

$$(23)(24)(25)(34)(35)$$
$$\times (1)^3(1+2+3)(1+2+3+4+5).$$

More generally,

$$\Big(\prod_{F\in R} x_F\Big) \prod_{j=1}^{|R|} \sum_{i=1}^{1+(d^T)_j} x_i,$$

where R = facets of $\mathrm{lk}_\Delta 1$ and d is the degree sequence of $\mathrm{del}_\Delta 1$.