Counting weighted simplicial spanning trees of shifted complexes

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COUNTING SPANNING TREES OF K_n

Thm (Cayley): K_n has n^{n-2} spanning trees.

T spanning tree: set of edges containing all vertices and

1. $|T| = n-1$
2. no cycles ($\tilde{H}_1(T) = 0$)
3. connected ($\tilde{H}_0(T) = 0$)

Note: Any two conditions imply the third.

WITH WHAT WEIGHTING?

vertices Silly ($n^{n-2}(x_1 \cdots x_n)$)

edges No nice structure (can't see n^{n-2})

edges and vertices Get Prüfer coding

$$\operatorname{wt} T = \prod_{e \in T} \operatorname{wt} e = \prod_{e \in T} \left(\prod_{v \in e} x_v \right)$$

$\sum_{T \in ST(K_n)} \operatorname{wt} T =$

$$(x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2}$$

ARBITRARY GRAPHS

Thm (Matrix Tree): Graph G has $|\det L_r(G)|$ spanning trees, where $L_r(G)$ is the reduced Laplacian matrix of G.

Defn 1: $L(G) = D(G) - A(G)$

$D(G) = \mathrm{diag}(\deg v_1, \ldots, \deg v_n)$

$A(G) =$ adjacency matrix

Defn 2: $L(G) = \partial(G)\partial(G)^T$

$\partial(G) =$ incidence matrix (boundary matrix)

"Reduced": remove rows/columns corresponding to any one vertex

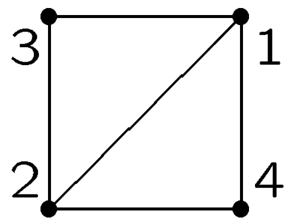
Proof (Matrix Tree Theorem):

$$\det L_r(G) = \det \partial_r(G)\partial_r(G)^T = \sum_T (\det \partial_r(T))^2$$
$$= \sum_T (\pm 1)^2$$

by Binet-Cauchy

EXAMPLES

Example:



$$\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Example: K_n

$$L(K_n) = nI - J \qquad (n \times n);$$
$$L_r(K_n) = nI - J \qquad (n-1 \times n-1)$$

$$\begin{aligned} \det L_r &= \prod \text{eigenvalues} \\ &= (n-0)^{(n-1)-1}(n-(n-1)) \\ &= n^{n-2} \end{aligned}$$

WEIGHTED MATRIX TREE THEOREM

$$\sum_{T \in ST(G)} \operatorname{wt} T = |\det \hat{L}_r(G)|,$$

where $\hat{L}$ is weighted Laplacian.

Defn 1: $\hat{L}(G) = \hat{D}(G) - \hat{A}(G)$

$D(G) = \operatorname{diag}(\hat{\deg} v_1, \ldots, \hat{\deg} v_n)$

$\hat{\deg} v_i = \sum_{v_i v_j \in E} x_i x_j$

$A(G) =$ adjacency matrix

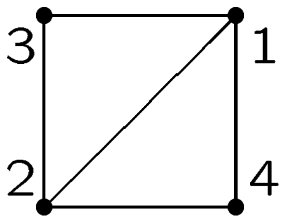
(entry $x_i x_j$ for edge $v_i v_j$)

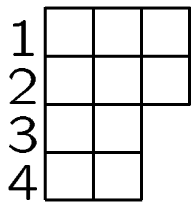
Defn 2: $\hat{L}(G) = \partial(G) B(G) \partial(G)^T$

$\partial(G) =$ incidence matrix

$B(G)$ diagonal, indexed by edges,

entry $\pm x_i x_j$ for edge $v_i v_j$

Example: 



$$\widehat{L} = \begin{pmatrix} 1(2+3+4) & -12 & -13 & -14 \\ -12 & 2(1+3+4) & -23 & -24 \\ -13 & -23 & 3(1+2) & 0 \\ -14 & -24 & 0 & 4(1+2) \end{pmatrix}$$

$$\widehat{L}_r = \begin{pmatrix} 2(1+3+4) & -23 & -24 \\ -23 & 3(1+2) & 0 \\ -24 & 0 & 4(1+2) \end{pmatrix}$$

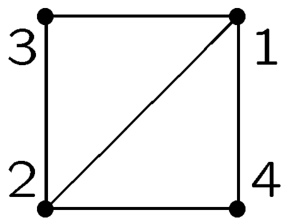
$$\begin{aligned}
\det \widehat{L}_r &= 234 \begin{vmatrix} 1+3+4 & -\sqrt{23} & -\sqrt{24} \\ -\sqrt{23} & 1+2 & 0 \\ -\sqrt{24} & 0 & 1+2 \end{vmatrix} \\
&= 234 \left| 1I + \begin{pmatrix} 3+4 & -\sqrt{23} & -\sqrt{24} \\ -\sqrt{23} & 2 & 0 \\ -\sqrt{24} & 0 & 2 \end{pmatrix} \right| \\
&= 234(1+0)(1+2)(1+2+3+4) \\
&= (1234)(1+2)(1+2+3+4)
\end{aligned}$$

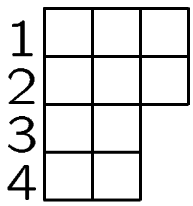
THRESHOLD GRAPHS

Defn 1: $V = 1, \ldots, n$
$E \in \mathcal{E}, i \notin E, j \in E, i < j \Rightarrow E \cup i - j \in \mathcal{E}$.

Defn 2: Can build recursively, by adding isolated vertices, and coning.

Ex:





Thm (Merris '94): Threshold graph has

$$\prod_{r \neq 1} (d^T)_r$$

spanning trees, where d is degree sequence.

Thm (Martin-Reiner '03; implied by Remmel-Williamson '02): If G is threshold, then

$$\sum_{T \in ST(G)} \mathrm{wt}\, T = (x_1 \cdots x_n) \prod_{r \neq 1} \Big(\sum_{i=1}^{(d^T)_r} x_i \Big).$$

SIMPLICIAL SPANNING TREES of K_n^k

Simplicial complex: $\Delta \subseteq 2^V$;
$F \subseteq G \in \Delta \Rightarrow F \in \Delta$.
K_n^k denotes the k-dimensional complete complex on n vertices (so $K_n = K_n^1$).

Simplicial spanning trees of K_n^k (Kalai, '83):
Set T, of k-dimensional faces, containing all $(k-1)$-dimensional faces and:

1. $|T| = \binom{n-1}{k}$
2. $\tilde{H}_k(T) = 0$
3. $\tilde{H}_{k-1}(T)$ is finite group

Note: Any two conditions imply the third.

How many are there?
Bolker ('76) should be $n^{\binom{n-2}{k}}$, but isn't

Kalai ('83) proves

$$\sum_{T \in SST(K_n^k)} |\tilde{H}_{k-1}(T)|^2 = n^{\binom{n-2}{k}}$$

WEIGHTING

As before, weight tree by product of the faces of the tree, and, for nice factoring, weight face by product of vertices.

$$\operatorname{wt} T = \prod_{F \in T} \operatorname{wt} F = \prod_{F \in T} \Big(\prod_{v \in F} x_v \Big)$$

Thm (Kalai '83):

$$\sum_{T \in SST(K_n)} |\tilde{H}_{k-1}(T)|^2 (\operatorname{wt} T)$$
$$= (x_1 \cdots x_n)^{\binom{n-2}{k-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{k}}$$

Adin ('92) did something similar for complete r-partite complexes.

KALAI'S THEOREM

Proof (unweighted; weighted is similar):

$$\begin{aligned} n^{\binom{n-2}{k}} &= \det L_r(K_n^k) = \det \partial_r(K_n^k)\partial_r(K_n^k)^T \\ &= \sum_T (\det \partial_r(T))^2 \\ &= \sum_T |\tilde{H}_{k-1}(T)|^2 \end{aligned}$$

by Binet-Cauchy, again.

"Reduced" now means pick one vertex, and then remove rows/columns corresponding to all $(k-1)$-dimensional faces containing that vertex.

$L = \partial\partial^T$

$\partial\colon \Delta_k \to \Delta_{k-1}$ boundary
$\partial^T\colon \Delta_{k-1} \to \Delta_k$ coboundary

EXAMPLE $n = 4, k = 2$

$$\partial^T = \begin{array}{c|cccccc} & 12 & 13 & 14 & 23 & 24 & 34 \\ \hline 123 & -1 & 1 & 0 & -1 & 0 & 0 \\ 124 & -1 & 0 & 1 & 0 & -1 & 0 \\ 134 & 0 & -1 & 1 & 0 & 0 & -1 \\ 234 & 0 & 0 & 0 & -1 & 1 & -1 \end{array}$$

$$L = \begin{pmatrix} 2 & -1 & -1 & 1 & 1 & 0 \\ -1 & 2 & -1 & -1 & 0 & 1 \\ -1 & -1 & 2 & 0 & -1 & -1 \\ 1 & -1 & 0 & 2 & -1 & 1 \\ 1 & 0 & -1 & -1 & 2 & -1 \\ 0 & 1 & -1 & 1 & -1 & 2 \end{pmatrix}$$

Note that $L_{F,G} \neq 0$ iff F and G differ by just one vertex.

SIMPLICIAL SPANNING TREES of ARBITRARY COMPLEXES

Defn: (Assume $\dim \Delta = k$.) k-dimensional complex T containing all $(k-1)$-dimensional faces of Δ ($T^{(k-1)} = \Delta^{(k-1)}$) and:

1. $f_k(T) = f_k(\Delta) - \beta_k(\Delta) + \beta_{k-1}(\Delta)$
2. $\tilde{H}_k(T) = 0$
3. $\tilde{H}_{k-1}(T)$ is finite group

Note: Any two conditions imply the third.

f_k is number of k-dimensional faces;
$\beta_k = \dim_{\mathbb{Q}} \tilde{H}_k$

SIMPLICIAL MATRIX TREE THEOREM

Thm (DKM):

$$\sum_{T \in SST(\Delta)} |\tilde{H}_{k-1}(T)|^2 = \frac{|\tilde{H}_{k-2}(\Delta)|^2}{|\tilde{H}_{k-2}(\Delta_U)|^2} \det L_r.$$

U =set of facets of $(k-1)$-spanning tree of Δ
L_r is L reduced by all of U
$\Delta_U = U \cup \Delta^{(k-2)}$

There is also analogous weighted version.

SHIFTED SIMPLICIAL COMPLEXES

Defn: $V = 1, \ldots, n$
$F \in \Delta, i \notin F, j \in F, i < j \Rightarrow F \cup i - j \in \Delta$ (equivalently, the k-faces form an initial ideal in the componentwise partial order).

Ex: bipyramid = $\{$123, 124, 125, 134, 135, 234, 235$\}$ (and subfaces)

$$\Delta = (1 * \mathrm{lk}_\Delta 1) \,\dot\cup\, B_\Delta$$

$B_\Delta = \{F \in \Delta \colon 1 \notin F, F \,\dot\cup\, 1 \notin \Delta\}$
$\mathrm{lk}_\Delta 1 = \{F - 1 \colon 1 \in F, F \in \Delta\}$, shifted
$\mathrm{del}_\Delta 1 = \{G \colon 1 \notin G, G \in \Delta\}$, shifted
$\beta_i(\Delta) = f_i(B_\Delta)$
$\mathrm{del}_\Delta 1 = \mathrm{lk}_\Delta 1 \,\dot\cup\, B_\Delta$

(D-Reiner '02) Eigenvalues of top-dimensional Laplacian given by d^T where d is degree sequence, $d_i = |\{\text{facets } F \colon i \in F\}|$.

EXAMPLE: BIPYRAMID

$B_\triangle = 234, 235$
$\mathrm{lk}_\triangle\, 1 = 23, 24, 25, 34, 25; 2, 3, 4, 5; \emptyset$
$\mathrm{del}_\triangle\, 1 = \mathrm{lk}_\triangle\, 1 \,\dot{\cup}\, B_\triangle$.
Eigenvalues

COUNTING TREES OF SHIFTED COMPLEXES

Pick U to be ridges ($(k-1)$-dimensional faces) containing 1, which is acyclic, and contains $(k-1)$-faces of Δ, and so it is (the facets of) a simplicial spanning tree.

Also, $\tilde{H}_{k-2}(\Delta_U) = 0$, and if Δ is pure and shifted, then $\tilde{H}_{k-2}(\Delta) = 0$, so we just have to compute $\det \hat{L}_r$.

Ex: bipyramid. Set of all ridges is all possible edges, except 45. $U = \{12, 13, 14, 15\}$, so $\hat{L}_r$ is indexed by $\{23, 24, 25, 34, 35\}$.

$$\hat{L}_r =$$

$$\begin{array}{ccccc}
23(1+4+5) & -234 & -235 & 234 & 235 \\
-234 & 24(1+3) & 0 & -234 & 0 \\
-235 & 0 & 25(1+3) & 0 & -235 \\
234 & 234 & 0 & (1+2)34 & 0 \\
235 & 0 & -235 & 0 & (1+2)35
\end{array}$$

SIMPLIFICATIONS

$$\det \hat{L}_r = (23)(24)(25)(34)(35) \det M$$

$$\det M = \begin{vmatrix} 1+4+5 & -\sqrt{34} & -\sqrt{35} & \sqrt{24} & \sqrt{25} \\ -\sqrt{34} & 1+3 & 0 & -\sqrt{23} & 0 \\ -\sqrt{35} & 0 & 1+3 & 0 & -\sqrt{23} \\ \sqrt{24} & -\sqrt{23} & 0 & 1+2 & 0 \\ \sqrt{25} & 0 & -\sqrt{23} & 0 & 1+2 \end{vmatrix}$$

$$= \left| 1I + N \right|$$

$$N = \begin{pmatrix} 4+5 & -\sqrt{34} & -\sqrt{35} & \sqrt{24} & \sqrt{25} \\ -\sqrt{34} & 3 & 0 & -\sqrt{23} & 0 \\ -\sqrt{35} & 0 & 3 & 0 & -\sqrt{23} \\ \sqrt{24} & -\sqrt{23} & 0 & 2 & 0 \\ \sqrt{25} & 0 & -\sqrt{23} & 0 & 2 \end{pmatrix}$$

Remarkably, N is a weighted Laplacian of $\mathrm{del}_\triangle 1$; $N = \partial\partial^T$ with

$$\partial^T = \begin{pmatrix} \sqrt{4} & -\sqrt{3} & 0 & \sqrt{2} & 0 \\ \sqrt{5} & 0 & -\sqrt{3} & 0 & \sqrt{2} \end{pmatrix}$$

The (F,G) entry of this matrix is $\pm\sqrt{F-G}$.

FURTHER SIMPLIFICATIONS

This makes the eigenvalues of this N be $0,0,0,2+3,2+3+4+5$, and $\det M = (1)^3(1+2+3)(1+2+3+4+5)$. Finally it makes the weighted tree enumerator of the bipyramid

$$(23)(24)(25)(34)(35) \\ \times (1)^3(1+2+3)(1+2+3+4+5).$$

More generally,

$$\Big(\prod_{F \in R} x_F\Big) \prod_{r=1}^{|R|} \sum_{i=1}^{1+d_r^T} x_i,$$

where $R =$ facets of $\mathrm{lk}_\Delta 1$ and d is the degree sequence of $\mathrm{del}_\Delta 1$.