Counting weighted simplicial spanning trees of shifted complexes

CombinaTexas

Texas A&M

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Art Duval
University of Texas at El Paso

Caroline Klivans
University of Chicago

Jeremy Martin
University of Kansas
COUNTING SPANNING TREES OF $K_n$

**Thm (Cayley):** $K_n$ has $n^{n-2}$ spanning trees.

$T$ spanning tree: set of edges containing all vertices and
1. $|T| = n - 1$
2. no cycles ($\tilde{H}_1(T) = 0$)
3. connected ($\tilde{H}_0(T) = 0$)
Note: Any two conditions imply the third.

**WITH WHAT WEIGHTING?**

**vertices:** Silly ($n^{n-2}(x_1 \cdots x_n)$)

**edges:** No nice structure (can’t see $n^{n-2}$)

**edges and vertices:** Get Prüfer coding

$$\text{wt } T = \prod_{e \in T} \text{wt } e = \prod_{e \in T} \left(\prod_{v \in e} x_v\right)$$

$$\sum_{T \in ST(K_n)} \text{wt } T =$$

$$(x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2}$$
ARBITRARY GRAPHS

Thm (Matrix Tree): Graph $G$ has $|\det L_r(G)|$ spanning trees, where $L_r(G)$ is the reduced Laplacian matrix of $G$.

Defn 1: $L(G) = D(G) - A(G)$

$D(G) = \text{diag}(\deg v_1, \ldots, \deg v_n)$

$A(G)$ = adjacency matrix

Defn 2: $L(G) = \partial(G)\partial(G)^T$  

$\partial(G)$ = incidence matrix (boundary matrix)

"Reduced": remove rows/columns corresponding to any one vertex

Proof (Matrix Tree Theorem):

$\det L_r(G) = \det \partial_r(G)\partial_r(G)^T = \sum_T (\det \partial_r(T))^2$

$= \sum_T (\pm 1)^2$

by Binet-Cauchy
EXAMPLES

Example:

\[
\begin{pmatrix}
3 & -1 & -1 & -1 \\
-1 & 3 & -1 & -1 \\
-1 & -1 & 2 & 0 \\
-1 & -1 & 0 & 2
\end{pmatrix}
, \begin{pmatrix}
-1 & -1 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & -1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1
\end{pmatrix}
\]

Example: \( K_n \)

\[ L(K_n) = nI - J \quad \text{\((n \times n)\)}; \]
\[ L_r(K_n) = nI - J \quad \text{\((n-1 \times n-1)\)} \]

\[ \text{det } L_r = \prod \text{eigenvalues} \]
\[ = (n - 0)^{(n-1) - 1}(n - (n-1)) \]
\[ = n^{n-2} \]
WEIGHTED MATRIX TREE THEOREM

\[ \sum_{T \in ST(G)} \text{wt } T = |\det \hat{L}_r(G')|, \]

where \( \hat{L} \) is weighted Laplacian.

Defn 1: \( \hat{L}(G') = \hat{D}(G) - \hat{A}(G) \)

\[ D(G') = \text{diag}(\hat{\text{deg}}v_1, \ldots, \hat{\text{deg}}v_n) \]
\[ \hat{\text{deg}}v_i = \sum_{v_iv_j \in E} x_i x_j \]

\( A(G) = \) adjacency matrix
(entry \( x_i x_j \) for edge \( v_iv_j \))

Defn 2: \( \hat{L}(G) = \partial(G') B(G) \partial(G)^T \)

\( \partial(G) = \) incidence matrix
\( B(G') \) diagonal, indexed by edges,
entry \( \pm x_i x_j \) for edge \( v_iv_j \)
Example:

\[
\begin{array}{c}
3 \\
2 \\
1 \\
4
\end{array}
\quad
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 4 \\
\hline
\end{array}
\]

\[\hat{L} = \begin{pmatrix}
1(2 + 3 + 4) & -12 & -13 & -14 \\
-12 & 2(1 + 3 + 4) & -23 & -24 \\
-13 & -23 & 3(1 + 2) & 0 \\
-14 & -24 & 0 & 4(1 + 2)
\end{pmatrix}\]

\[\hat{L}_r = \begin{pmatrix}
2(1 + 3 + 4) & -23 & -24 \\
-23 & 3(1 + 2) & 0 \\
-24 & 0 & 4(1 + 2)
\end{pmatrix}\]

\[\det \hat{L}_r = 234 \begin{vmatrix}
1 + 3 + 4 & -\sqrt{23} & -\sqrt{24} \\
-\sqrt{23} & 1 + 2 & 0 \\
-\sqrt{24} & 0 & 1 + 2
\end{vmatrix}\]

\[= 234 \begin{vmatrix}
1 + 0 \\
-\sqrt{23} & 2 & 0 \\
-\sqrt{24} & 0 & 2
\end{vmatrix}\]

\[= 234(1 + 0)(1 + 2)(1 + 2 + 3 + 4)\]

\[= (1234)(1 + 2)(1 + 2 + 3 + 4)\]
THRESHOLD GRAPHS

Defn 1: \( V = 1, \ldots, n \)
\[ E \in \mathcal{E}, i \notin E, j \in E, i < j \Rightarrow E \cup i - j \in \mathcal{E}. \]

Defn 2: Can build recursively, by adding isolated vertices, and coning.

Ex: 

\[
\begin{array}{ccc}
& 3 & \\
2 & 1 & 4 \\
\end{array}
\quad 
\begin{array}{cccc}
1 & & & 2 \\
& 1 & 3 & \\
& & 4 & \end{array}
\]

Thm (Merris ‘94): Threshold graph has

\[
\prod_{r \neq 1} (d^T)_r
\]
spanning trees, where \( d \) is degree sequence.

Thm (Martin-Reiner ‘03; implied by Remmel-Williamson ‘02): If \( G \) is threshold, then

\[
\sum_{T \in ST(G)} \text{wt} \ T = (x_1 \cdots x_n) \prod_{r \neq 1} (\sum_{i=1}^{(d^T)_r} x_i).
\]
SIMPLICIAL SPANNING TREES of $K_n^k$

Simplicial complex: $\Delta \subseteq 2^V$;
$F \subseteq G \in \Delta \Rightarrow F \in \Delta$.
$K_n^k$ denotes the $k$-dimensional complete complex on $n$ vertices (so $K_n = K_n^1$).

Simplicial spanning trees of $K_n^k$ (Kalai, ‘83):
Set $T$, of $k$-dimensional faces, containing all $(k - 1)$-dimensional faces and:
1. $|T| = \binom{n-1}{k}$
2. $\tilde{H}_k(T) = 0$
3. $\tilde{H}_{k-1}(T)$ is finite group
Note: Any two conditions imply the third.

How many are there?
Bolker (‘76) should be $n^{(n-2)}$, but isn’t
Kalai (‘83) proves
$$\sum_{T \in \text{SST}(K_n^k)} |\tilde{H}_{k-1}(T)|^2 = n^{(n-2)}$$
WEIGHTING

As before, weight tree by product of the faces of the tree, and, for nice factoring, weight face by product of vertices.

\[ \text{wt } T = \prod_{F \in T} \text{wt } F = \prod_{F \in T} \left( \prod_{v \in F} x_v \right) \]

Thm (Kalai ’83):

\[ \sum_{T \in \text{SST}(K_n)} |\hat{H}_{k-1}(T)|^2(\text{wt } T) = (x_1 \cdots x_n)^{\binom{n-2}{k-1}}(x_1 + \cdots + x_n)^{\binom{n-2}{k}} \]

Adin (’92) did something similar for complete \( r \)-partite complexes.
KALAI’S THEOREM

Proof (unweighted; weighted is similar):

\[ n^{(n-2)} = \det L_r(K^k_n) = \det \partial_r(K^k_n) \partial_r(K^k_n)^T \]
\[ = \sum T (\det \partial_r(T))^2 \]
\[ = \sum T |\tilde{H}_{k-1}(T)|^2 \]

by Binet-Cauchy, again.

“Reduced” now means pick one vertex, and then remove rows/columns corresponding to all \((k-1)\)-dimensional faces containing that vertex.

\[ L = \partial \partial^T \]

\(\partial: \Delta_k \rightarrow \Delta_{k-1}\) boundary
\(\partial^T: \Delta_{k-1} \rightarrow \Delta_k\) coboundary
EXAMPLE $n = 4, k = 2$

$$
\delta^T = \begin{array}{c|ccccccc}
123 & 12 & 13 & 14 & 23 & 24 & 34 \\
124 & -1 & 1 & 0 & -1 & 0 & 0 \\
134 & -1 & 0 & 1 & 0 & -1 & 0 \\
234 & 0 & -1 & 1 & 0 & 0 & -1 \\
\end{array}
$$

$$
L = \begin{pmatrix}
2 & -1 & -1 & 1 & 1 & 0 \\
-1 & 2 & -1 & -1 & 0 & 1 \\
-1 & -1 & 2 & 0 & -1 & -1 \\
1 & -1 & 0 & 2 & -1 & 1 \\
1 & 0 & -1 & -1 & 2 & -1 \\
0 & 1 & -1 & 1 & -1 & 2 \\
\end{pmatrix}
$$

Note that $L_{F,G} \neq 0$ iff $F$ and $G$ differ by just one vertex.
**SIMPLICIAL SPANNING TREES of ARBITRARY COMPLEXES**

Defn: (Assume $\dim \Delta = k$.) $k$-dimensional complex $T$ containing all $(k - 1)$-dimensional faces of $\Delta$ ($T^{(k-1)} = \Delta^{(k-1)}$) and:

1. $f_k(T) = f_k(\Delta) - \beta_k(\Delta) + \beta_{k-1}(\Delta)$

2. $\tilde{H}_k(T) = 0$

3. $\tilde{H}_{k-1}(T')$ is finite group

Note: Any two conditions imply the third.

$f_k$ is number of $k$-dimensional faces;
$\beta_k = \dim_{\mathbb{Q}} \tilde{H}_k$
SIMPLICIAL MATRIX TREE THEOREM

Thm (DKM):

\[ \sum_{T \in \text{SST} (\Delta)} |\tilde{H}_{k-1}(T)|^2 = \frac{|\tilde{H}_{k-2}(\Delta)|^2}{|\tilde{H}_{k-2}(\Delta_U)|^2} \det L_r. \]

\( U = \) set of facets of \((k-1)\)-spanning tree of \( \Delta \)

\( L_r \) is \( L \) reduced by all of \( U \)

\( \Delta_U = U \cup \Delta^{(k-2)} \)

There is also analogous weighted version.
Defn: \( V = 1, \ldots, n \)
\( F \in \Delta, i \not\in F, j \in F, i < j \Rightarrow F \cup i - j \in \Delta \)
(equivalently, the \( k \)-faces form an initial ideal in the componentwise partial order).

Ex: bipyramid = \{123, 124, 125, 134, 135, 234, 235\} (and subfaces)
\[ \Delta = (1 \star \text{lk}_\Delta 1) \cup B_\Delta \]

\( B_\Delta = \{F \in \Delta : 1 \not\in F, F \cup 1 \not\in \Delta\} \)
\( \text{lk}_\Delta 1 = \{F - 1 : 1 \in F, F \in \Delta\} \), shifted
\( \text{del}_\Delta 1 = \{G : 1 \not\in G, G \in \Delta\} \), shifted
\( \beta_i(\Delta) = f_i(B_\Delta) \)
\( \text{del}_\Delta 1 = \text{lk}_\Delta 1 \cup B_\Delta \)

(D-Reiner ‘02) Eigenvalues of top-dimensional Laplacian given by \( d^T \) where \( d \) is degree sequence, \( d_i = |\{\text{facets } F' : i \in F\}|. \)
EXAMPLE: BIPYRAMID

$B_\Delta = 234, 235$

$\text{lk}_\Delta 1 = 23, 24, 25, 34, 25; 2, 3, 4, 5; \emptyset$

$\text{del}_\Delta 1 = \text{lk}_\Delta 1 \cup B_\Delta$.

Eigenvalues
COUNTING TREES OF SHIFTED COMPLEXES

Pick $U$ to be ridges ($(k-1)$-dimensional faces) containing 1, which is acyclic, and contains $(k-1)$-faces of $\Delta$, and so it is (the facets of) a simplicial spanning tree.

Also, $\tilde{H}_{k-2}(\Delta_U) = 0$, and if $\Delta$ is pure and shifted, then $\tilde{H}_{k-2}(\Delta) = 0$, so we just have to compute $\det \hat{L}_r$.

Ex: bipyramid. Set of all ridges is all possible edges, except 45. $U = \{12, 13, 14, 15\}$, so $\hat{L}_r$ is indexed by $\{23, 24, 25, 34, 35\}$.

\[
\hat{L}_r = \\
\begin{array}{cccccc}
23(1 + 4 + 5) & -234 & -235 & 234 & 235 \\
-234 & 24(1 + 3) & 0 & -234 & 0 \\
-235 & 0 & 25(1 + 3) & 0 & -235 \\
234 & 234 & 0 & (1 + 2)34 & 0 \\
235 & 0 & -235 & 0 & (1 + 2)35 \\
\end{array}
\]
SIMPLIFICATIONS

\[
\det \hat{L}_r = (23)(24)(25)(34)(35) \det M
\]

\[
\begin{vmatrix}
1 + 4 + 5 & -\sqrt{34} & -\sqrt{35} & \sqrt{24} & \sqrt{25} \\
-\sqrt{34} & 1 + 3 & 0 & -\sqrt{23} & 0 \\
-\sqrt{35} & 0 & 1 + 3 & 0 & -\sqrt{23} \\
\sqrt{24} & -\sqrt{23} & 0 & 1 + 2 & 0 \\
\sqrt{25} & 0 & -\sqrt{23} & 0 & 1 + 2 \\
\end{vmatrix}
\]

\[
= \begin{vmatrix} 1 \end{vmatrix} + N
\]

\[
N = \begin{pmatrix}
4 + 5 & -\sqrt{34} & -\sqrt{35} & \sqrt{24} & \sqrt{25} \\
-\sqrt{34} & 3 & 0 & -\sqrt{23} & 0 \\
-\sqrt{35} & 0 & 3 & 0 & -\sqrt{23} \\
\sqrt{24} & -\sqrt{23} & 0 & 2 & 0 \\
\sqrt{25} & 0 & -\sqrt{23} & 0 & 2 \\
\end{pmatrix}
\]

Remarkably, \( N \) is a weighted Laplacian of \( \text{del}_\Delta 1 \);
\( N = \partial \partial^T \) with

\[
\partial^T = \begin{pmatrix}
\sqrt{4} & -\sqrt{3} & 0 & \sqrt{2} & 0 \\
\sqrt{5} & 0 & -\sqrt{3} & 0 & \sqrt{2} \\
\end{pmatrix}
\]

The \((F, G)\) entry of this matrix is \( \pm \sqrt{F - G} \).
FURTHER SIMPLIFICATIONS

This makes the eigenvalues of this \( N \) be 
\( 0, 0, 0, 2 + 3, 2 + 3 + 4 + 5 \), and 
\[ \det M = (1)^3(1 + 2 + 3)(1 + 2 + 3 + 4 + 5). \]
Finally it makes the weighted tree enumerator of the bipyramid

\[
(23)(24)(25)(34)(35) \\
\times (1)^3(1 + 2 + 3)(1 + 2 + 3 + 4 + 5).
\]

More generally,

\[
\left( \prod_{F \in R} x_F \right) \prod_{r=1}^{|R|} \sum_{i=1}^{1+d_r^T} x_i,
\]

where \( R = \text{facets of } \text{lk}_\Delta 1 \) and \( d \) is the degree sequence of \( \text{del}_\Delta 1 \).