Max flow min cut in higher dimensions

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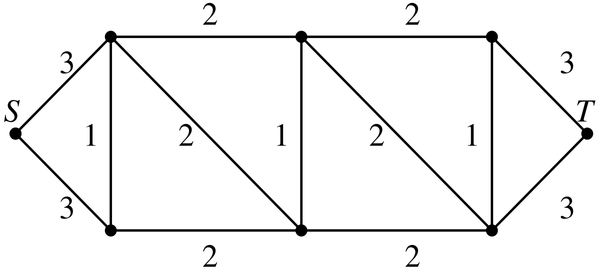
Max flow



Given a graph G with source S, sink T, and edge capacities κ.

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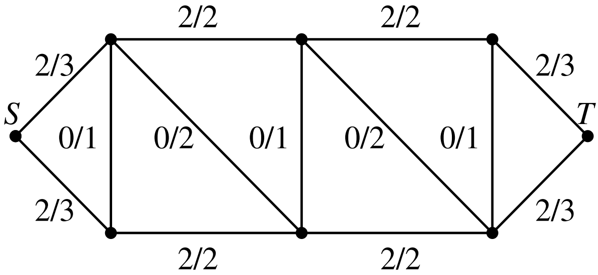

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Definition

Flow on G is an assignment of flow x_e (non-negative number, and direction) to each edge such that:

- net flow at each vertex, except S and T, is zero; and
- $|x_e| \leq \kappa_e$.

Value of flow is outflow(S) = inflow(T).

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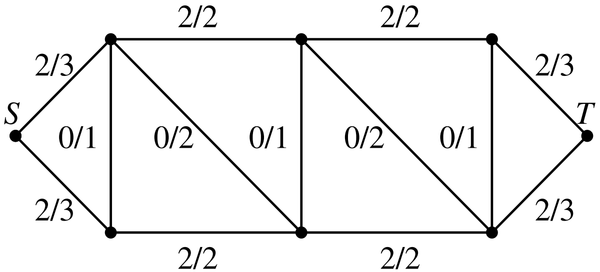

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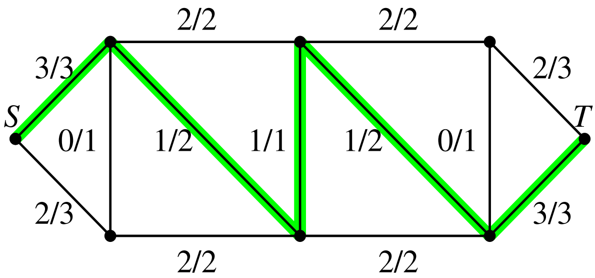

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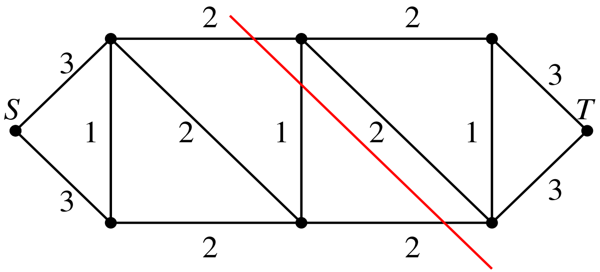

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Cut is minimal set of edges whose removal disconnects S from T.
Value of cut is $\sum_{e \in \text{cut}} \kappa_e$.

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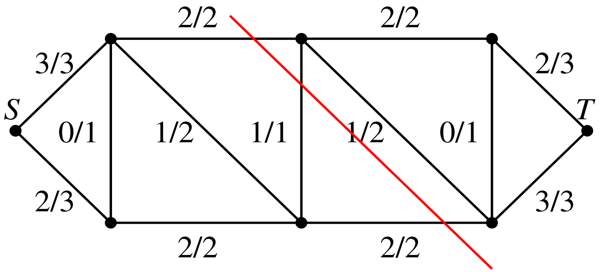

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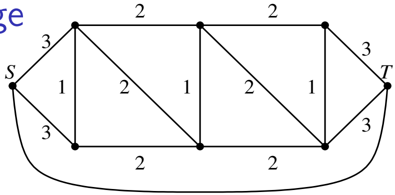
Clearly, value(flow) $\leq$ value(cut), so max flow $\leq$ min cut.

Theorem (Classic max flow min cut)

Max flow = min cut.

Add an extra edge



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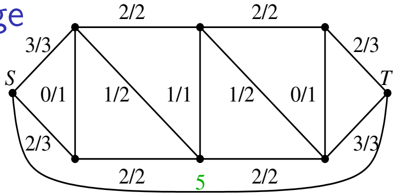

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Value of flow is x_0.

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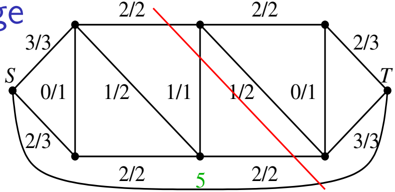

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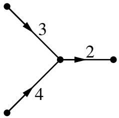
Cut is minimal set of edges, including e_0, whose removal disconnects G. Value of cut is $\sum_{e \in \text{cut} \setminus e_0} \kappa_e$.

Flows and boundary



$$\begin{pmatrix}1 & 1 & -1\end{pmatrix}\begin{pmatrix}3\\4\\2\end{pmatrix} = 5.$$

Assign orientation to each edge (flow going “backwards” gets negative value)

$$\mathrm{netflow}(v) = \sum_{v=e^+} x_e - \sum_{v=e^-} x_e = \sum_{v\in e}(-1)^{\varepsilon(e,v)} x_e = (\partial x)_v$$

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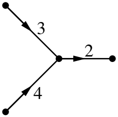

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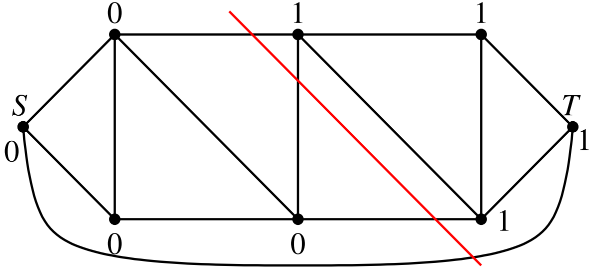
So net flow condition is $\partial x = 0$.

Cuts and coboundary

Assign 1 to every vertex in connected component with T, 0 to others. Let y_v be value at v. Edges in cut are those that have both 0 and 1 endpoints.



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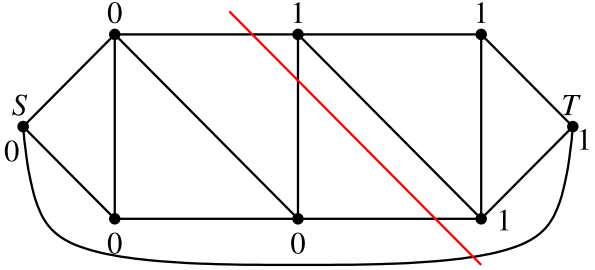

Coboundary will do this: $\partial^T y$ (linear combination of rows of ∂) gives characteristic vector of cut.

Linear programming

Flow is now a linear program

- Find vector x (in edge space)
- $\partial x = 0$ (x is in flow space)
- $-\kappa_e \leq x_e \leq \kappa_e$ (can omit e_0)
- max x_0

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The dual program is (can easily be reworked to say):

- Find vector y (in vertex space)
- Let $u = \partial^T y$ (in cut space)
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Linear programming says the solutions are equal; with some effort we can show the solution to the dual LP is the min cut problem.

Max flow in higher dimensions

Example: 2-dimensional complex; ∂ maps 2-dimensional cells (polygons) to edges.

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- identify designated polygon p_0; max x_0

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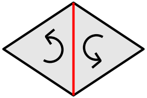
Find a 1-dimensional cycle on the complex, and attach a polygon face filling that cycle. We are trying to maximize circulation on that designated polygon (around that cycle), while making all circulation balance on each edge.



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Linear programming says the solutions are equal; with some effort we can show the solution to the dual LP is the min cut problem. In particular, the support on this cut is a minimal set of polygons whose removal introduces codimension-1 homology, e.g., 1-dimensional "circular" hole in 2-dimensional complex

Summary

Theorem (DKM)

The max circulation around a codimension-1 cycle (e.g., 1-dimensional cycle in 2-dimensional complex) equals the value of a minimum cut containing the added face that fills in the cycle (e.g., polygon filling in 1-dimensional cycle).

Fine print:

- cut is minimal set of faces whose removal increases codimension-1 homology
- cut vector is in span of row space of boundary matrix
- normalize cut vector by specifying its value is 1 on p_0, the added filling-in face
- cut vector might not be all 1's and 0's
- value of cut is inner product of capacities with cut vector