Weighted spanning tree enumerators of complete colorful complexes

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Spanning trees of $K_n$

Theorem (Cayley)

$K_n$ has $n^{n-2}$ spanning trees.

$T \subseteq E(G)$ is a spanning tree of $G$ when:

0. spanning: $T$ contains all vertices;
1. connected ($\tilde{H}_0(T) = 0$)
2. no cycles ($\tilde{H}_1(T) = 0$)
3. correct count: $|T| = n - 1$

If 0. holds, then any two of 1., 2., 3. together imply the third condition.
Theorem (Cayley-Prüfer)

\[
\sum_{T \in ST(K_n)} \text{wt } T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2},
\]

where \( \text{wt } T = \prod_{e \in T} \text{wt } e = \prod_{e \in T} (\prod_{v \in e} x_v). \)
Theorem (Cayley-Prüfer)

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Example \((K_4)\)
Theorem (Cayley-Prüfer)

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Example (\( K_4 \))

- 4 trees like: \( T = \begin{array}{c}
3 \\
2 \\
1 \\
4
\end{array} \)

\( \text{wt } T = (x_1x_2x_3x_4)x_2^2 \)
Theorem (Cayley-Prüfer)

\[ \sum_{T \in ST(K_n)} \text{wt } T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2}, \]

where \( \text{wt } T = \prod_{e \in T} \text{wt } e = \prod_{e \in T} (\prod_{v \in e} x_v). \)

Example \((K_4)\)

- 4 trees like: \( T = \)

\[ \begin{array}{c}
\text{2} \\
\text{3} \\
\text{4} \\
\text{1}
\end{array} \]

\( \text{wt } T = (x_1 x_2 x_3 x_4) x_2^2 \)

- 12 trees like: \( T = \)

\[ \begin{array}{c}
\text{2} \\
\text{3} \\
\text{1} \\
\text{4}
\end{array} \]

\( \text{wt } T = (x_1 x_2 x_3 x_4) x_1 x_3 \)
Theorem (Cayley-Prüfer)

\[ \sum_{T \in ST(K_n)} \text{wt } T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2}, \]

where \( \text{wt } T = \prod_{e \in T} \text{wt } e = \prod_{e \in T} (\prod_{v \in e} x_v). \)

Example \((K_4)\)

- 4 trees like: \( T = \)

- 12 trees like: \( T = \)

- Total is \( (x_1 x_2 x_3 x_4)(x_1 + x_2 + x_3 + x_4)^2. \)
Complete bipartite graphs

Example \((K_{3,2})\)
Complete bipartite graphs

Example $\mathcal{K}_{3,2}$

6 trees like: $T =$

```
1
2
3
```

$\mathrm{wt\ }T = (12312)12^2$

Total is $(12312)(1 + 2 + 3)(1 + 2)^2$. 

Theorem

$$\sum_{T \in \text{ST}(\mathcal{K}_m, n)} \mathrm{wt\ }T = (x_1 \cdots x_m)(y_1 \cdots y_n)(x_1 + \cdots + x_m)^{n-1}(y_1 + \cdots + y_n)^{m-1}.$$ 

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Complete bipartite graphs

Example \((K_{3,2})\)

- 6 trees like: \(T = \begin{array}{c} 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \end{array} \) \text{ wt } T = (12312)12^2

- 6 trees like: \(T = \begin{array}{c} 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \end{array} \) \text{ wt } T = (12312)212
Complete bipartite graphs

Example \((K_{3,2})\)

- 6 trees like: \(T = \begin{array}{c}
1 \\
2 \\
3
\end{array} \begin{array}{c}
1 \\
2 \\
2
\end{array} \) \(\text{wt } T = (12312)12^2\)

- 6 trees like: \(T = \begin{array}{c}
1 \\
2 \\
3
\end{array} \begin{array}{c}
1 \\
2 \\
2
\end{array} \) \(\text{wt } T = (12312)212\)

- Total is \((12312)(1 + 2 + 3)(1 + 2)^2\).
Complete bipartite graphs

Example \((K_{3,2})\)

- 6 trees like: \(T = \begin{array}{c}
1 \\
2 \\
3 \\
1 \\
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3 \\
\end{array}\) \(\text{wt } T = (12312)12^2\)

- 6 trees like: \(T = \begin{array}{c}
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3 \\
\end{array}\) \(\text{wt } T = (12312)212\)

- Total is \((12312)(1 + 2 + 3)(1 + 2)^2\).

Theorem

\[
\sum_{T \in ST(K_{m,n})} \text{wt } T = (x_1 \cdots x_m)(y_1 \cdots y_n)(x_1 + \cdots + x_m)^{n-1}(y_1 + \cdots + y_n)^{m-1}.
\]
Laplacian

Theorem (Kirchoff’s Matrix-Tree)

$G$ has $|\text{det } L_r(G)|$ spanning trees

Definition The Laplacian matrix of graph $G$, denoted by $L(G)$. 

"Reduced": remove rows/columns corresponding to any one vertex
Theorem (Kirchoff’s Matrix-Tree)

\( G \) has \( |\det L_r(G)| \) spanning trees

Definition The Laplacian matrix of graph \( G \), denoted by \( L(G) \).

Defn 1: \( L(G) = D(G) - A(G) \)

\( D(G) = \text{diag}(\deg v_1, \ldots, \deg v_n) \)

\( A(G) = \text{adjacency matrix} \)
Laplacian

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\( A(G) = \text{adjacency matrix} \)

Defn 2: \( L(G) = \partial(G)\partial(G)^T \)

\( \partial(G) = \text{incidence matrix (boundary matrix)} \)
Laplacian

**Theorem (Kirchoff’s Matrix-Tree)**

$G$ has $\det L_r(G)$ spanning trees

**Definition** The reduced Laplacian matrix of graph $G$, denoted by $L_r(G)$.

Defn 1: $L(G) = D(G) - A(G)$

- $D(G) = \text{diag}(\deg v_1, \ldots, \deg v_n)$
- $A(G) = \text{adjacency matrix}$

Defn 2: $L(G) = \partial(G)\partial(G)^T$

- $\partial(G) = \text{incidence matrix (boundary matrix)}$

“Reduced”: remove rows/columns corresponding to any one vertex
Example \((K_{3,2})\)

\[
\begin{align*}
1 & \quad 2 & \quad 3 \\
\end{align*}
\]

\[
\partial = \begin{pmatrix}
1 & -1 & -1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & -1 & -1 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & -1 & -1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 \\
2 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
2 & 0 & 0 & -1 & -1 \\
0 & 2 & 0 & -1 & -1 \\
0 & 0 & 2 & -1 & -1 \\
-1 & -1 & -1 & 3 & 0 \\
-1 & -1 & -1 & 0 & 3 \\
\end{pmatrix}
\]

\[
\text{det}(L) = 12, \text{ the number of spanning trees of } K_{3,2}.
\]
Example \((K_{3,2})\)

\[
\begin{pmatrix}
2 & 0 & 0 & -1 & -1 \\\n0 & 2 & 0 & -1 & -1 \\
0 & 0 & 2 & -1 & -1 \\
-1 & -1 & -1 & 3 & 0 \\
-1 & -1 & -1 & 0 & 3
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
11 & 12 & 21 & 22 & 31 & 32 \\
1 & -1 & -1 & 0 & 0 & 0 \\
2 & 0 & 0 & -1 & -1 & 0 \\
3 & 0 & 0 & 0 & 0 & -1 & -1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 \\
2 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}
\]

\[
\partial = \begin{pmatrix}
-1 & 0 & -1 \\
-1 & -1 & 0 \\
-1 & -1 & -1 \\
-1 & -1 & 3 \\
-1 & -1 & 0 \\
-1 & -1 & 0 \\
\end{pmatrix}
\]

\[
\text{det}(L_r) = 12, \text{ the number of spanning trees of } K_{3,2}.
\]
Example \((K_{3,2})\)

\[
\begin{align*}
L &= \begin{pmatrix}
2 & 0 & 0 & -1 & -1 \\
0 & 2 & 0 & -1 & -1 \\
0 & 0 & 2 & -1 & -1 \\
-1 & -1 & -1 & 3 & 0 \\
-1 & -1 & -1 & 0 & 3 \\
\end{pmatrix},
L_r &= \begin{pmatrix}
2 & 0 & -1 & -1 \\
0 & 2 & -1 & -1 \\
-1 & -1 & 3 & 0 \\
-1 & -1 & 0 & 3 \\
\end{pmatrix}
\end{align*}
\]

\[
\det(L_r) = 12, \text{ the number of spanning trees of } K_{3,2}.
\]
Weighted Matrix-Tree Theorem

\[ \sum_{T \in ST(G)} \text{wt } T = | \det \hat{L}_r(G) |, \]

where \( \hat{L}_r(G) \) is reduced weighted Laplacian.

Defn 1: \( \hat{L}(G) = \hat{D}(G) - \hat{A}(G) \)

\[ \hat{D}(G) = \text{diag}(\hat{\deg} v_1, \ldots, \hat{\deg} v_n) \]
\[ \hat{\deg} v_i = \sum_{v_i v_j \in E} x_i x_j \]
\[ \hat{A}(G) = \text{adjacency matrix} \]
(entry \( x_i x_j \) for edge \( v_i v_j \))

Defn 2: \( \hat{L}(G) = \partial(G) B(G) \partial(G)^T \)

\[ \partial(G) = \text{incidence matrix} \]
\[ B(G) \text{ diagonal, indexed by edges,} \]
(entry \( \pm x_i x_j \) for edge \( v_i v_j \))
Example \((K_{3,2})\)

\[
\hat{L}_r = \begin{pmatrix}
2(1 + 2) & 0 & -21 & -22 \\
0 & 3(1 + 2) & -31 & -32 \\
-21 & -31 & 1(1 + 2 + 3) & 0 \\
-22 & -32 & 0 & 2(1 + 2 + 3)
\end{pmatrix}
\]

\[
\det \hat{L}_r = (12312)(1 + 2 + 3)(1 + 2)^2
\]
Simplicial spanning trees of $K_n^d$ [Kalai, ’83]

Let $K_n^d$ denote the complete $d$-dimensional simplicial complex on $n$ vertices. $\Upsilon \subseteq K_n^d$ is a **simplicial spanning tree** of $K_n^d$ when:

0. $\Upsilon_{(d-1)} = K_n^{d-1}$ (“spanning”);
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group (“connected”);
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ (“acyclic”);
3. $|\Upsilon| = \binom{n-1}{d}$ (“count”).

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.
Counting simplicial spanning trees of $K_n^d$

**Conjecture** [Bolker ’76]

\[
\sum_{\Upsilon \in \text{SST}(K_n^d)} = n \binom{n-2}{d}
\]
Counting simplicial spanning trees of $K_n^d$

**Theorem** [Kalai ’83]

$$
\tau(K_n^d) = \sum_{\Upsilon \in \text{SST}(K_n^d)} |\tilde{H}_{d-1}(\Upsilon)|^2 = n\binom{n-2}{d}
$$
Weighted simplicial spanning trees of $K^d_n$

As before,

$$\text{wt } \Upsilon = \prod_{F \in \Upsilon} \text{wt } F = \prod_{F \in \Upsilon} \left( \prod_{v \in F} x_v \right)$$

Example

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$

$$\text{wt } \Upsilon = x_1^5 x_2^4 x_3^3 x_4^3 x_5$$
Weighted simplicial spanning trees of $K^d_n$

As before,

$$\text{wt } \gamma = \prod_{F \in \gamma} \text{wt } F = \prod_{F \in \gamma} \left( \prod_{v \in F} x_v \right)$$

Example

$$\gamma = \{123, 124, 125, 134, 135, 245\}$$

$$\text{wt } \gamma = x_1^5 x_2^4 x_3^3 x_4 x_5^3$$

Theorem (Kalai, '83)

$$\hat{\tau}(K^d_n) := \sum_{T \in \text{SST}(K^d_n)} |\tilde{H}_{d-1}(\gamma)|^2 (\text{wt } \gamma)$$

$$= (x_1 \cdots x_n)^{\binom{n-2}{d-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{d}}$$
Proof

Proof uses determinant of reduced Laplacian of $K_n^d$. “Reduced” now means pick one vertex, and then remove rows/columns corresponding to all $(d-1)$-dimensional faces containing that vertex.

$L = \partial \partial^T$

$\partial : \Delta_d \rightarrow \Delta_{d-1}$ boundary

$\partial^T : \Delta_{d-1} \rightarrow \Delta_d$ coboundary

Weighted version: Multiply column $F$ of $\partial$ by $x_F$
Example $n = 4, d = 2$ (tetrahedron)

$$\partial^T = \begin{bmatrix}
12 & 13 & 14 & 23 & 24 & 34 \\
123 & -1 & 1 & 0 & -1 & 0 & 0 \\
124 & -1 & 0 & 1 & 0 & -1 & 0 \\
134 & 0 & -1 & 1 & 0 & 0 & -1 \\
234 & 0 & 0 & 0 & -1 & 1 & -1
\end{bmatrix}$$

$$L = \begin{pmatrix}
2 & -1 & -1 & 1 & 1 & 0 \\
-1 & 2 & -1 & -1 & 0 & 1 \\
-1 & -1 & 2 & 0 & -1 & -1 \\
1 & -1 & 0 & 2 & -1 & 1 \\
1 & 0 & -1 & -1 & 2 & -1 \\
0 & 1 & -1 & 1 & -1 & 2
\end{pmatrix}$$

$$\det L_r = 4$$
Simplicial spanning trees of arbitrary simplicial complexes

Let $\Delta$ be a $d$-dimensional simplicial complex. $\Upsilon \subseteq \Delta$ is a simplicial spanning tree of $\Delta$ when:

0. $\Upsilon_{(d-1)} = \Delta_{(d-1)}$ (“spanning”);
1. $\tilde{\mathcal{H}}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group (“connected”);
2. $\tilde{\mathcal{H}}_d(\Upsilon; \mathbb{Z}) = 0$ (“acyclic”);
3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$ (“count”).

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.
Simplicial Matrix-Tree Theorem

Theorem (D.-Klivans-Martin, ’09)

\[ \hat{\tau}(\Delta) = \frac{|\tilde{H}_{d-2}(\Delta; \mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma; \mathbb{Z})|^2} \det \hat{L}_\Gamma, \]

where

- \( \Gamma \in \text{SST}(\Delta_{(d-1)}) \)
- \( \partial_\Gamma = \text{restriction of } \partial_d \text{ to faces not in } \Gamma \)
- \( \text{reduced Laplacian } L_\Gamma = \partial_\Gamma \partial^T_\Gamma \)
- \( \text{Weighted version: Multiply column } F \text{ of } \partial \text{ by } x_F \)

Note: The \( |\tilde{H}_{d-2}| \) terms are often trivial.
Example: Octahedron

- Vertices 1, 2, 1, 2, 1, 2.
- Facets 111, 112, 121, 122, 211, 212, 221, 222,
- $\Gamma = 11, 12, 11, 12, 22$ spanning tree of 1-skeleton, so remove (rows and columns corresponding to) those edges from weighted Laplacian.
- $\det \hat{L}_\Gamma = \begin{pmatrix} 12 & 12 & 12 \end{pmatrix}^3 (1 + 2)(1 + 2)(1 + 2)$. 
Complete colorful complexes

Definition (Adin, ’92)
The complete colorful complex $K_{n_1,\ldots,n_r}$ is a simplicial complex with:

▶ vertex set $V_1 \dot\cup \ldots \dot\cup V_r$ ($V_i$ is set of vertices of color $i$);
▶ $|V_i| = n_i$;
▶ faces are all sets of vertices with no repeated colors.

Example
Octahedron is $K_{222}$.
Theorem (Adin, ’92)

The top-dimensional spanning trees of $K_{n_1,...,n_r}$ are “counted” by

\[ \tau(K_{n_1,...,n_r}) = \prod_{i=1}^{r} n_i \prod_{j \neq i} (n_j - 1). \]

Note: Adin also has a more general formula for dimension less than $r - 1$.

Example

- $\tau(K_{222}) = 2^1 \times 2^1 \times 2^1$
- $\tau(K_{235}) = 2^{2 \cdot 4} \times 3^{1 \cdot 4} \times 5^{1 \cdot 2}$
- $\tau(K_{m,n}) = m^{n-1} \times n^{m-1}$
Weighted enumeration

Theorem (Aalipour-D.)

The top-dimensional spanning trees of $K_{n_1,\ldots,n_r}$ are “counted” by

$$\tau(K_{n_1,\ldots,n_r}) = \prod_{i=1}^{r}(x_{i,1} + \cdots + x_{i,n_i})\prod_{j\neq i}(n_j-1)(x_{i,1} \cdots x_{i,n_i})(\prod_{j\neq i} n_j)-(\prod_{j\neq i}(n_j-1)).$$

Example

$$\hat{\tau}(K_{235}) = (x_1 + x_2)^{2.4}(x_1x_2)^{3.5-2.4}$$

$$\times (y_1 + y_2 + y_3)^{1.4}(y_1y_2y_3)^{2.5-1.4}$$

$$\times (z_1 + \cdots + z_5)^{1.2}(z_1 \cdots z_5)^{2.3-1.2}$$
Codimension-1 spanning tree (Adin)

We will use the weighted simplicial matrix-tree theorem. So first we have to find a codimension-1 spanning tree. But it will be a different tree for each color. For each color’s factors, treat that color as “last”.
Codimension-1 spanning tree (Adin)

We will use the weighted simplicial matrix-tree theorem. So first we have to find a codimension-1 spanning tree. But it will be a different tree for each color. For each color’s factors, treat that color as “last”.

$r = 3$ (1-dimensional spanning tree): Start with 1, and attach to every other vertex, except blue vertices. Then use 1 to connect the remaining blue vertices.

$r = 4$ (2-dimensional spanning tree): Start with 1, and attach to every edge with no blue vertices. Then use 1, and attach to all edges using a blue non-1 vertex with a non-red vertex. Finally use 1 with edges with a blue non-1 vertex with a red non-1 vertex.
Codimension-1 spanning tree (Adin)

We will use the weighted simplicial matrix-tree theorem. So first we have to find a codimension-1 spanning tree. But it will be a different tree for each color. For each color’s factors, treat that color as “last”.

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$r = 4$ (2-dimensional spanning tree): Start with 1, and attach to every edge with no blue vertices. Then use 1, and attach to all edges using a blue non-1 vertex with a non-red vertex. Finally use 1 with edges with a blue non-1 vertex with a red non-1 vertex.
Final thought

Terry Pratchett, *The Colour of Magic*: “Do you not know that what you belittle by the name tree is but the mere four-dimensional analogue of a whole multidimensional universe which...”
Final thought

Terry Pratchett, *The Colour of Magic*:
“Do you not know that what you belittle by the name *tree* is but the mere four-dimensional analogue of a whole multidimensional universe which—no, I can see you do not.”
Final thought

Terry Pratchett, *The Colour of Magic*:
“Do you not know that what you belittle by the name *tree* is but the mere four-dimensional analogue of a whole multidimensional universe which—no, I can see you do not.”

But, now, *you* do.