A non-partitionable Cohen-Macaulay simplicial complex

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Partitionability Conjecture

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*No.*
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Stanley: “I am glad that this problem has finally been put to rest, though I would have preferred a proof rather than a counterexample. Perhaps you can withdraw your paper from the arXiv and come up with a proof instead.”
Simplicial complexes

Definition (Simplicial complex)
Let $V$ be set of vertices. Then $\Delta$ is a simplicial complex on $V$ if:
- $\Delta \subseteq 2^V$; and
- if $\sigma \subseteq \tau \in \Delta$ implies $\tau \in \Delta$.

Higher-dimensional analogue of graph.
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Higher-dimensional analogue of graph.

**Definition ($f$-vector)**

$f_i = f_i(\Delta) =$ number of $i$-dimensional faces of $\Delta$. The $f$-vector of $d$-dimensional $\Delta$ is

$$f(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_d)$$
Example

124, 125, 134, 135, 234, 235;
12, 13, 14, 15, 23, 24, 25, 34, 35;
1, 2, 3, 4, 5;
∅

\[ f(\Delta) = (1, 5, 9, 6) \]
Counting faces of spheres

**Definition (Sphere)**
Simplicial complex whose realization is a triangulation of a sphere.

**Conjecture (Upper Bound)**
*Explicit upper bound on* \( f_i \) *of a sphere with given dimension and number of vertices.*

This was proved by Stanley in 1975. Some of the key ingredients:

- face-ring (algebraic object derived from the simplicial complex) [Stanley, Hochster]
- face-ring of sphere is Cohen-Macaulay [Reisner]
Cohen-Macaulay simplicial complexes

CM rings of great interest in commutative algebra (depth = dimension). Here is a more topological/combinatorial definition.

Definition (Link)
\[ \text{lk}_\Delta \sigma = \{ \tau \in \Delta : \tau \cap \sigma = \emptyset, \ \tau \cup \sigma \in \Delta \} , \text{ what } \Delta \text{ looks like near } \sigma. \]
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Definition (Homology)
\[ \tilde{H}_i(\Delta) = \ker \partial_i / \text{im} \partial_{i+1}, \text{ measures } i\text{-dimensional “holes” of } \Delta. \]
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**Definition (Homology)**
\[ \tilde{H}_i(\Delta) = \ker \partial_i / \text{im} \partial_{i+1}, \] measures \( i \)-dimensional “holes” of \( \Delta \).

**Theorem (Reisner ’76)**
*Face-ring of \( \Delta \) is Cohen-Macaulay if, for all \( \sigma \in \Delta \),*

\[ \tilde{H}_i(\text{lk}_\Delta \sigma) = 0 \quad \text{for } i < \dim \text{lk}_\Delta \sigma. \]

We take this as our definition of CM simplicial complex.
Cohen-Macaulayness is topological

Recall our definition:

**Theorem (Reisner ’76)**

*Face-ring of $\Delta$ is *Cohen-Macaulay* if, for all $\sigma \in \Delta$,*

\[
\tilde{H}_i(\text{lk}_\Delta \sigma) = 0 \quad \text{for } i < \dim \text{lk}_\Delta \sigma.
\]

Munkres (’84) showed that CM is a *topological* condition. That is, it only depends on (the homeomorphism class of) the *realization* of $\Delta$. In particular, spheres and balls are CM.

**Example**

\[
\begin{array}{c}
\cap \\
\text{is not CM}
\end{array}
\]
**h-vector**

The conditions for the UBC most easily stated in terms of \(h\)-vector.

**Definition (\(h\)-vector)**

Let \( \dim \Delta = d \).

\[
h_k = h_k(\Delta) = \sum_{j=0}^{k} (-1)^{k-j} \binom{d+1-j}{k-j} f_{j-1}, \quad 0 \leq k \leq d + 1.
\]

Equivalently,

\[
\sum_{i=-1}^{d} f_i t^{d-i} = \sum_{k=0}^{d+1} h_k (t+1)^{d+1-k}
\]

The \(h\)-vector of \(\Delta\) is \(h(\Delta) = (h_0, h_1, \ldots, h_{d+1})\).
Example

\[ f(\Delta) = (1, 5, 9, 6), \text{ and} \]

\[ 1t^3 + 5t^2 + 9t + 6 = 1(t + 1)^3 + 2(t + 1)^2 + 2(t + 1)^1 + 1 \]

so \( h(\Delta) = (1, 2, 2, 1). \)
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so \( h(\Delta) = (1, 2, 2, 1). \)

Note that in this case, \( h \geq 0. \) This is a consequence of the algebraic defn of CM. But how could we see this combinatorially?
**Partitionability**

\[ 1t^3 + 5t^2 + 9t + 6 = 1(t + 1)^3 + 2(t + 1)^2 + 2(t + 1)^1 + 1 \]

A non-partitionable Cohen-Macaulay simplicial complex
Partitionability

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**Definition (Partitionable)**

When a simplicial complex can be partitioned like this, into Boolean intervals whose tops are facets, we say the complex is partitionable.
Shellability

Most CM complexes in combinatorics are shellable:

**Definition (Shellable)**

A simplicial complex is *shellable* if it can be built one facet at a time, so that there is always a unique new face being added. A shelling is a particular kind of partitioning.
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**Proposition**

If $\Delta$ is shellable, then $h_k$ counts number of intervals whose bottom (the unique new face) is dimension $k - 1$.

**Example**

In our previous example, minimal new faces were: $\emptyset$, vertex, edge, vertex, edge, triangle.
We were trying to prove the conjecture

Idea of our “proof”:

- Remove all the faces containing a given vertex (this will be the first part of the partitioning).

Example

A non-partitionable Cohen-Macaulay simplicial complex
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- Remove all the faces containing a given vertex (this will be the first part of the partitioning).
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Example

Duval, Goeckner, Klivans, Martin
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Idea of our “proof”:

▶ Remove all the faces containing a given vertex (this will be the first part of the partitioning).
▶ Try to make sure what’s left is relative CM.
▶ Apply induction.

The problem is we would have to prove the conjecture for relative CM complexes.

Example

Duval, Goeckner, Klivans, Martin

A non-partitionable Cohen-Macaulay simplicial complex
Relative simplicial complexes

Definition (Relative simplicial complex)

Φ is a relative simplicial complex on $V$ if:

- $\Phi \subseteq 2^V$; and
- $\rho \subseteq \sigma \subseteq \tau$ and $\rho, \tau \in \Phi$ together imply $\sigma \in \Phi$

We can write any relative complex $\Phi$ as $\Phi = (\Delta, \Gamma)$, for some pair of simplicial complexes $\Gamma \subseteq \Delta$.

Example

\[ \text{A non-partitionable Cohen-Macaulay simplicial complex} \]
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- Φ ⊆ 2^V; and
- ρ ⊆ σ ⊆ τ and ρ, τ ∈ Φ together imply σ ∈ Φ

We can write any relative complex Φ as Φ = (∆, Γ), for some pair of simplicial complexes Γ ⊆ ∆. But ∆ and Γ are not unique.

Example

A non-partitionable Cohen-Macaulay simplicial complex
Relative Cohen-Macaulay

Recall we take

\[ \tilde{H}_i(\text{lk}_\Delta \sigma) = 0 \quad \text{for } i < \text{lk}_\Delta \sigma \]

as our definition of CM simplicial complex.
Relative Cohen-Macualay

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as our definition of CM simplicial complex. This generalizes easily:

**Theorem (Stanley ’87)**

*Face-ring of $\Phi = (\Delta, \Gamma)$ is relative Cohen-Macaulay if, for all $\sigma \in \Delta$,

$$\tilde{H}_i(\text{lk}_\Delta \sigma, \text{lk}_\Gamma \sigma) = 0 \quad \text{for } i < \text{lk}_\Delta \sigma.$$*
Relative Cohen-Macaulay

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**Example**

A non-partitionable Cohen-Macaulay simplicial complex
Looking for a non-trivial example

Still trying to prove conjecture:

- We wanted to find a non-trivial example of something Cohen-Macaulay and partitionable, so we could see how this idea of relative complexes would work.
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Still trying to prove conjecture:

- We wanted to find a non-trivial example of something Cohen-Macaulay and partitionable, so we could see how this idea of relative complexes would work.
- How hard is it to take that second step of the partitioning, which is the first step for the relative complex?
- Idea: non-trivial = not shellable; CM = ball (and if it’s not partitionable, we’re done). So we are looking for non-shellable balls.
M.E. Rudin’s non-shellable ball

First we tried M.E. Rudin’s (’58) non-shellable 3-ball:

- 3-dimensional (built out of tetrahedra);
- 14 vertices;
- 41 tetrahedra;
- Can be realized as triangulation of tetrahedron with all vertices on boundary.

Did not help.
Ziegler’s non-shellable ball

Next we tried Ziegler’s (’98) non-shellable 3-ball with 10 vertices and 21 tetrahedra
Ziegler’s non-shellable ball

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Just because it is partitionable does not mean you can start partitioning in any order. So we started to partition until we could not go any further (without backtracking). This part uses the computer!
First pass with Ziegler

Recall: Searching through Ziegler’s non-shellable ball, by partitioning greedily, until you can’t. We found a relative complex:

- 6 vertices, 5 facets, but 4 triangles on boundary removed
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Proposition

If $X$ and $(X, A)$ are CM and $\dim A = \dim X - 1$, then gluing together two copies of $X$ along $A$ gives a CM (non-relative) complex.
Pigeonhole principle

Recall our example \((X, A)\) is:

- relative Cohen-Macaulay
- not partitionable

If we glue together two copies of \(X\) along \(A\), is it partitionable?
Pigeonhole principle

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If we glue together two copies of \(X\) along \(A\), is it partitionable? Maybe: some parts of \(A\) can help partition one copy of \(X\), other parts of \(A\) can help partition the other copy of \(X\).
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**Remark**
If we glue together many copies of \(X\) along \(A\), at least one copy will be missing all of \(A\)!
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If we glue together many copies of \(X\) along \(A\), at least one copy will be missing all of \(A\)! How many is enough?
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Remark

If we glue together many copies of \(X\) along \(A\), at least one copy will be missing all of \(A\)! How many is enough? More than the number of all faces in \(A\). Then the result will not be partitionable.
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**Remark**

If we glue together many copies of \(X\) along \(A\), at least one copy will be missing all of \(A\)! How many is enough? More than the number of all faces in \(A\). Then the result will not be partitionable. But the resulting complex is not actually a simplicial complex because of repeats.
Induced subcomplexes

To avoid this problem, we need to make sure that $A$ is vertex-induced. This means every face in $X$ among vertices in $A$ must be in $A$ as well. (Minimal faces of $(X, A)$ are vertices.)
Induced subcomplexes

To avoid this problem, we need to make sure that $A$ is vertex-induced. This means every face in $X$ among vertices in $A$ must be in $A$ as well. (Minimal faces of $(X, A)$ are vertices.) To summarize, this what we need:

- $X$, $(X, A)$ relative CM
- $A$ vertex-induced (minimal faces of $(X, A)$ are vertices)
- $(X, A)$ not partitionable
Eureka!

By computer search, we found that if

- $Z$ is Ziegler’s 3-ball, and
- $B = Z$ restricted to all vertices except 1,5,9 ($B$ has 7 facets),

then $Q = (Z, B)$ satisfies all our criteria!
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then $Q = (Z, B)$ satisfies all our criteria!

Also $Q = (X, A)$, where $X$ has 14 facets, and $A$ is 5 triangles:

$$
\begin{align*}
1249 & \quad 1269 \\
1569 & \quad 1589 \\
1489 & \quad 1458 \\
1457 & \quad 4578 \\
1256 & \quad 0125 \\
0256 & \quad 0123 \\
1234 & \quad 1347
\end{align*}
$$
Putting it all together

- Since $A$ has 24 faces total (including the empty face), we know gluing together 25 copies of $X$ along their common copy of $A$, the resulting (non-relative) complex is CM, not partitionable.
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- In fact, computer search showed that gluing together only 3 copies of $X$ will do it. Resulting complex has $f$-vector $(1, 16, 71, 98, 42)$. 

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Putting it all together

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► In fact, computer search showed that gluing together only 3 copies of $X$ will do it. Resulting complex has $f$-vector $(1, 16, 71, 98, 42)$.

► Later we found short proof by hand to show that 3 copies of $X$ suffices.
Stanley depth (a brief summary)

Definition (Stanley)
If $I$ is a monomial ideal in a polynomial ring $S$, then the Stanley depth $sdepth S/I$ is a purely combinatorial analogue of depth, defined in terms of certain vector space decompositions of $S/I$.

Conjecture (Stanley '82)
For all monomial ideals $I$, $sdepth S/I \geq \text{depth } S/I$.

Theorem (Herzog, Jahan, Yassemi '08)
If $I$ is the Stanley-Reisner ideal (related to the face ring) of a Cohen-Macaulay complex $\Delta$, then the inequality $sdepth S/I \geq \text{depth } S/I$ is equivalent to the partitionability of $\Delta$.

Corollary
Our counterexample disproves this conjecture as well.
Constructibility

Definition

A $d$-dimensional simplicial complex $\Delta$ is **constructible** if:

- it is a simplex; or
- $\Delta = \Delta_1 \cup \Delta_2$, where $\Delta_1, \Delta_2, \Delta_1 \cap \Delta_2$ are constructible of dimensions $d, d, d - 1$, respectively.

Theorem

Constructible complexes are Cohen-Macaulay.

Question (Hachimori '00)

Are constructible complexes partitionable?

Corollary

Our counterexample is constructible, so the answer to this question is no.

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Smaller counterexample?

Open questions:

**Question**

*Is there a smaller 3-dimensional counterexample to the partitionability conjecture?*
Smaller counterexample?

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**Question**

*Is the partitionability conjecture true in 2 dimensions?*
Save the conjecture: Strengthen the hypothesis

More open questions (based on what our counterexample is not): Note that our counterexample is not a ball (3 balls sharing common 2-dimensional faces), but all balls are CM.

Question

Are simplicial balls partitionable?
Save the conjecture: Strengthen the hypothesis

More open questions (based on what our counterexample is not): Note that our counterexample is not a ball (3 balls sharing common 2-dimensional faces), but all balls are CM.

Question
Are simplicial balls partitionable?

Definition (Balanced)
A simplicial complex is balanced if vertices can be colored so that every facet has one vertex of each color.

Question
Are balanced Cohen-Macaulay complexes partitionable?
Save the conjecture: Weaken the conclusion

**Question**

*What does the h-vector of a CM complex count?*
Save the conjecture: Weaken the conclusion

Question

*What does the $h$-vector of a CM complex count?*

One possible answer (D.-Zhang '01) replaces Boolean intervals with “Boolean trees”. But maybe there are other answers.