Simplicial spanning trees

Art Duval\textsuperscript{1} \hspace{0.5cm} Caroline Klivans\textsuperscript{2} \hspace{0.5cm} Jeremy Martin\textsuperscript{3}

\textsuperscript{1}University of Texas at El Paso
\textsuperscript{2}University of Chicago
\textsuperscript{3}University of Kansas

4th Joint UTEP/NMSU Workshop on Mathematics, Computer Science, and Computational Sciences
University of Texas at El Paso
November 8, 2008
Counting weighted spanning trees of $K_n$

**Theorem** [Cayley]: $K_n$ has $n^{n-2}$ spanning trees.

A spanning tree $T$ is a set of edges containing all vertices and:

1. connected ($\tilde{H}_0(T) = 0$)
2. no cycles ($\tilde{H}_1(T) = 0$)
3. $|T| = n - 1$

Note: Any two conditions imply the third.
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both! \[ \text{wt } T = \prod_{e \in T} \text{wt } e = \prod_{e \in T} \left( \prod_{v \in e} x_v \right) \] Prüfer coding

\[ \sum_{T \in ST(K_n)} \text{wt } T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2} \]
Example: $K_4$

4 trees like:

$T = (x_1x_2x_3x_4)x_2^2$
Example: $K_4$

- 4 trees like: $T = \begin{array}{c}
\text{1} \\
\text{3} \\
\text{2} \\
\end{array}$

  wt $T = (x_1x_2x_3x_4)x_2^2$

- 12 trees like: $T = \begin{array}{c}
\text{1} \\
\text{4} \\
\text{2} \\
\end{array}$

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Example: $K_4$

- 4 trees like: $T = \begin{align*}
\begin{array}{c}
3 \\
2 \\
3
\end{array}
\end{align*}$

  $\text{wt } T = (x_1 x_2 x_3 x_4) x_2^2$

- 12 trees like: $T = \begin{align*}
\begin{array}{c}
2 \\
4
\end{array}
\end{align*}$

  $\text{wt } T = (x_1 x_2 x_3 x_4) x_1 x_3$

Total is $(x_1 x_2 x_3 x_4)(x_1 + x_2 + x_3 + x_4)^2$. 
Laplacian

**Definition** The Laplacian matrix of graph $G$, denoted by $L(G)$. 

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Laplacian

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Defn 1: $L(G) = D(G) - A(G)$

- $D(G) = \text{diag} (\text{deg } v_1, \ldots, \text{deg } v_n)$
- $A(G) = \text{adjacency matrix}$
Laplacian

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Defn 2: $L(G) = \partial(G)\partial(G)^T$

- $\partial(G) = \text{incidence matrix (boundary matrix)}$
Laplacian

**Definition** The **reduced Laplacian** matrix of graph $G$, denoted by $L_r(G)$.

**Defn 1:** $L(G) = D(G) - A(G)$

$D(G) = \text{diag}(\text{deg } v_1, \ldots, \text{deg } v_n)$

$A(G) = \text{adjacency matrix}$

**Defn 2:** $L(G) = \partial(G)\partial(G)^T$

$\partial(G) = \text{incidence matrix (boundary matrix)}$

“Reduced”: remove rows/columns corresponding to any one vertex
Example

\[
\begin{array}{c}
3 \\
2 \\
1 \\
4
\end{array}
\]

\[
\partial = \begin{pmatrix}
1 & 12 & 13 & 14 & 23 & 24 \\
1 & -1 & -1 & -1 & 0 & 0 \\
2 & 1 & 0 & 0 & -1 & -1 \\
3 & 0 & 1 & 0 & 1 & 0 \\
4 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
3 & -1 & -1 & -1 \\
-1 & 3 & -1 & -1 \\
-1 & -1 & 2 & 0 \\
-1 & -1 & 0 & 2
\end{pmatrix}
\]
Matrix-Tree Theorems

**Version I** Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of $L$. Then $G$ has

$$\frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}$$

spanning trees.

**Version II** $G$ has $|\det L_r(G)|$ spanning trees

**Proof** [Version II]

$$\det L_r(G) = \det \partial_r(G)\partial_r(G)^T = \sum_T (\det \partial_r(T))^2$$

$$= \sum_T (\pm 1)^2$$

by Binet-Cauchy
Weighted Matrix-Tree Theorem

\[ \sum_{T \in ST(G)} \text{wt } T = | \det \hat{L}_r(G) | , \]

where \( \hat{L} \) is weighted Laplacian.

Defn 1: \( \hat{L}(G) = \hat{D}(G) - \hat{A}(G) \)

\( \hat{D}(G) = \text{diag}(\hat{\deg} v_1, \ldots, \hat{\deg} v_n) \)

\( \hat{\deg} v_i = \sum_{v_i v_j \in E} x_i x_j \)

\( \hat{A}(G) = \text{adjacency matrix} \)

(entry \( x_i x_j \) for edge \( v_i v_j \))

Defn 2: \( \hat{L}(G) = \partial(G)B(G)\partial(G)^T \)

\( \partial(G) = \text{incidence matrix} \)

\( B(G) \) diagonal, indexed by edges,

(entry \( \pm x_i x_j \) for edge \( v_i v_j \))

Duval, Klivans, Martin

Simplicial spanning trees
Example

\[
\hat{L} = \begin{pmatrix}
1(2 + 3 + 4) & -12 & -13 & -14 \\
-12 & 2(1 + 3 + 4) & -23 & -24 \\
-13 & -23 & 3(1 + 2) & 0 \\
-14 & -24 & 0 & 4(1 + 2)
\end{pmatrix}
\]

\[
\det \hat{L}_r = (1234)(1 + 2)(1 + 2 + 3 + 4)
\]
Threshold graphs

- Vertices 1, \ldots, n

Example

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{example_threshold_graph.png}
\caption{Example of a threshold graph.}
\end{figure}
Threshold graphs

- Vertices $1, \ldots, n$
- $E \in \mathcal{E}, i \notin E, j \in E, i < j \Rightarrow E \cup i - j \in \mathcal{E}$.

Example

![Graph example]
Threshold graphs

- Vertices 1, \ldots, n
- \( E \in \mathcal{E}, i \notin E, j \in E, i < j \Rightarrow E \cup i - j \in \mathcal{E} \).
- Equivalently, the edges form an initial ideal in the componentwise partial order.

Example

\begin{itemize}
  \item \begin{tikzpicture}[scale=0.8]
    \node (1) at (0,0) {1};
    \node (2) at (-1,-1) {2};
    \node (3) at (1,-1) {3};
    \node (4) at (0,-2) {4};
    \draw (1) -- (2); \draw (1) -- (3); \draw (1) -- (4);
  \end{tikzpicture}
  \item \begin{tikzpicture}[scale=0.8]
    \node (1) at (0,0) {1};
    \node (2) at (-1,-1) {2};
    \node (3) at (-1,-2) {3};
    \node (4) at (1,-1) {4};
    \node (5) at (1,-2) {5};
    \draw (1) -- (2); \draw (1) -- (3); \draw (1) -- (4); \draw (1) -- (5);
  \end{tikzpicture}
\end{itemize}
Weighted spanning trees of threshold graphs

**Theorem** [Martin-Reiner ‘03; implied by Remmel-Williamson ‘02]:
If $G$ is threshold, then

$$
\sum_{T \in ST(G)} \text{wt } T = (x_1 \cdots x_n) \prod_{r \neq 1} (\sum_{i=1}^{(d^T)_r} x_i).
$$

**Example**

$$
(1234)(1 + 2)(1 + 2 + 3 + 4)
$$
Complete skeleta of simplicial complexes

Simplicial complex $\Sigma \subseteq 2^V$; $F \subseteq G \in \Sigma \Rightarrow F \in \Sigma$. 

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Complete skeleta of simplicial complexes

Simplicial complex $\Sigma \subseteq 2^V$;
$F \subseteq G \in \Sigma \Rightarrow F \in \Sigma$.

Complete skeleton The $k$-dimensional complete complex on $n$ vertices, i.e.,

$$K^k_n = \{F \subseteq V : |F| \leq k + 1\}$$

(so $K_n = K^1_n$).
Simplicial spanning trees of $K_n^k$ [Kalai, ’83]

$\Upsilon \subseteq K_n^k$ is a **simplicial spanning tree** of $K_n^k$ when:

1. $\Upsilon_{(k-1)} = K_n^{k-1}$ (“spanning”);
2. $\tilde{H}_{k-1}(\Upsilon; \mathbb{Z})$ is a finite group (“connected”);
3. $\tilde{H}_k(\Upsilon; \mathbb{Z}) = 0$ (“acyclic”);
4. $|\Upsilon| = \binom{n-1}{k}$ (“count”).

▶ If 0. holds, then any two of 1., 2., 3. together imply the third condition.
▶ When $k = 1$, coincides with usual definition.
Counting simplicial spanning trees of $K_n^k$

**Conjecture** [Bolker ’76]

\[
\sum_{\Upsilon \in SST(K_n^k)} \left| \tilde{H}_{k-1}(\Upsilon) \right| = n^{n-2}\binom{n-2}{k}
\]
Counting simplicial spanning trees of $K_n^k$

**Theorem** [Kalai '83]

$$\sum_{\Upsilon \in SST(K_n^k)} |\tilde{H}_{k-1}(\Upsilon)|^2 = n^{n-2\choose k}$$
Weighted simplicial spanning trees of $K^k_n$

As before,

$$\text{wt } \Upsilon = \prod_{F \in \Upsilon} \text{wt } F = \prod_{F \in \Upsilon} \left( \prod_{v \in F} x_v \right)$$

Example:

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$

$$\text{wt } \Upsilon = x_1^5 x_2^4 x_3^3 x_4^3 x_5^3$$
Weighted simplicial spanning trees of $K_n^k$

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Example:

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$

$$\text{wt } \Upsilon = x_1^5 x_2^4 x_3^3 x_4^3 x_5^3$$

**Theorem** [Kalai, ’83]

$$\sum_{\Upsilon \in SST(K_n^k)} |\tilde{H}_{k-1}(\Upsilon)|^2(\text{wt } \Upsilon) = (x_1 \cdots x_n)^{(n-2)}(x_1 + \cdots + x_n)^{(n-k)}$$
Weighted simplicial spanning trees of $K_n^k$

As before,

$$\text{wt } \Upsilon = \prod_{F \in \Upsilon} \text{wt } F = \prod_{F \in \Upsilon} \left( \prod_{v \in F} x_v \right)$$

Example:

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$

$$\text{wt } \Upsilon = x_5 \cdot x_4 \cdot x_3 \cdot x_4 \cdot x_5$$

Theorem [Kalai, '83]

$$\sum_{\Upsilon \in \text{SST}(K_n)} |\tilde{H}_{k-1}(\Upsilon)|^2 (\text{wt } \Upsilon) = (x_1 \cdots x_n)^{\binom{n-2}{k-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{k-2}}$$

(Adin ('92) did something similar for complete $r$-partite complexes.)
Proof

Proof uses determinant of reduced Laplacian of $K_n^k$. “Reduced” now means pick one vertex, and then remove rows/columns corresponding to all $(k - 1)$-dimensional faces containing that vertex.

$L = \partial \partial^T$

$\partial : \Delta_k \rightarrow \Delta_{k-1}$ boundary

$\partial^T : \Delta_{k-1} \rightarrow \Delta_k$ coboundary

Weighted version: Multiply column $F$ of $\partial$ by $x_F$
Example $n = 4, k = 2$

\[\partial^T = \begin{array}{cccccc}
123 & 12 & 13 & 14 & 23 & 24 & 34 \\
   & -1 & 1 & 0 & -1 & 0 & 0 \\
124 & -1 & 0 & 1 & 0 & -1 & 0 \\
134 & 0 & -1 & 1 & 0 & 0 & -1 \\
234 & 0 & 0 & 0 & -1 & 1 & -1 \\
\end{array}\]

\[L = \begin{pmatrix}
2 & -1 & -1 & 1 & 1 & 0 \\
-1 & 2 & -1 & -1 & 0 & 1 \\
-1 & -1 & 2 & 0 & -1 & -1 \\
1 & -1 & 0 & 2 & -1 & 1 \\
1 & 0 & -1 & -1 & 2 & -1 \\
0 & 1 & -1 & 1 & -1 & 2 \\
\end{pmatrix}\]
Simplicial spanning trees of arbitrary simplicial complexes

Let Σ be a $d$-dimensional simplicial complex. $\Upsilon \subseteq \Sigma$ is a simplicial spanning tree of $\Sigma$ when:

0. $\Upsilon_{(d-1)} = \Sigma_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $f_d(\Upsilon) = f_d(\Sigma) - \tilde{\beta}_d(\Sigma) + \tilde{\beta}_{d-1}(\Sigma)$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.
Example

Bipyramid with equator, $\langle 123, 124, 125, 134, 135, 234, 235 \rangle$

- 6 SST’s not containing face 123
Example

Bipyramid with equator, $\langle 123, 124, 125, 134, 135, 234, 235 \rangle$

- 6 SST’s not containing face 123
- 9 SST’s containing face 123
Example

Bipyramid with equator, $\langle 123, 124, 125, 134, 135, 234, 235 \rangle$

- 6 SST’s not containing face 123
- 9 SST’s containing face 123

Total is $(x_1x_2x_3)^3(x_4x_5)^2(x_1 + x_2 + x_3)(x_1 + x_2 + x_3 + x_4 + x_5)$.
Simplicial Matrix-Tree Theorem — Version I

- $\Sigma$ a $d$-dimensional “metaconnected” simplicial complex
- $(d-1)$-dimensional (up-down) Laplacian $L_{d-1} = \partial_{d-1} \partial_{d-1}^T$
- $s_d = \text{product of nonzero eigenvalues of } L_{d-1}$.

**Theorem** [DKM]

$$h_d := \sum_{\gamma \in \text{SST}(\Sigma)} |\tilde{H}_{d-1}(\gamma)|^2 = \frac{s_d}{h_{d-1}} |\tilde{H}_{d-2}(\Sigma)|^2$$
Simplicial Matrix-Tree Theorem — Version II

\( \Gamma \in \text{SST}(\Sigma_{(d-1)}) \)
\( \partial \Gamma = \text{restriction of } \partial_d \text{ to faces not in } \Gamma \)
\( \text{reduced Laplacian } L_\Gamma = \partial_\Gamma \partial_{\Gamma}^* \)

**Theorem** [DKM]

\[
 h_d = \sum_{\gamma \in \text{SST}(\Sigma)} |\tilde{H}_{d-1}(\gamma)|^2 = \frac{|\tilde{H}_{d-2}(\Sigma; \mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma; \mathbb{Z})|^2} \det L_\Gamma.
\]

**Note:** The \( |\tilde{H}_{d-2}| \) terms are often trivial.
Weighted Simplicial Matrix-Tree Theorems

- Introduce an indeterminate $x_F$ for each face $F \in \Delta$
- Weighted boundary $\partial$: multiply column $F$ of (usual) $\partial$ by $x_F$
- $\partial_\Gamma = \text{restriction of } \partial_d \text{ to faces not in } \Gamma$
- Weighted reduced Laplacian $L_\Gamma = \partial_\Gamma \partial_\Gamma^*$

**Theorem** [DKM]

$$h_d := \sum_{\gamma \in SST(\Sigma)} |\tilde{H}_{d-1}(\gamma)|^2 \prod_{F \in \gamma} x_F^2 = \frac{s_d}{h_{d-1}} |\tilde{H}_{d-2}(\Sigma)|^2$$

$$h_d = \frac{|\tilde{H}_{d-2}(\Delta; \mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma; \mathbb{Z})|^2} \det L_\Gamma.$$
Definition of shifted complexes

- Vertices 1, \ldots, n
- \( F \in \Sigma, i \not\in F, j \in F, i < j \Rightarrow F \cup i - j \in \Sigma \)
- Equivalently, the \( k \)-faces form an initial ideal in the componentwise partial order.

**Example** (bipyramid with equator)
\( \langle 123, 124, 125, 134, 135, 234, 235 \rangle \)
Hasse diagram

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Simplicial spanning trees
Hasse diagram

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Simplicial spanning trees
Links and deletions

- Deletion, $\text{del}_1 \Sigma = \{ G : 1 \not\in G, G \in \Sigma \}$.
- Link, $\text{lk}_1 \Sigma = \{ F - 1 : 1 \in F, F \in \Sigma \}$.
- Deletion and link are each shifted, with vertices 2, \ldots, $n$.
- Example:

  $$\Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle$$
Links and deletions

- **Deletion**, $\text{del}_1 \Sigma = \{ G : 1 \not\in G, G \in \Sigma \}$.
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- Deletion and link are each shifted, with vertices $2, \ldots, n$.
- **Example**:

  $\Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle$

  $\text{del}_1 \Sigma = \langle 234, 235 \rangle$
Links and deletions

- **Deletion**, $\text{del}_1 \Sigma = \{ G : 1 \not\in G, G \in \Sigma \}$.
- **Link**, $\text{lk}_1 \Sigma = \{ F - 1 : 1 \in F, F \in \Sigma \}$.
- Deletion and link are each shifted, with vertices $2, \ldots, n$.
- **Example**:

  \[
  \Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle \\
  \text{del}_1 \Sigma = \langle 234, 235 \rangle \\
  \text{lk}_1 \Sigma = \langle 23, 24, 25, 34, 35 \rangle 
  \]
Weighted enumeration of SST’s in shifted complexes

**Theorem** Let \( \Lambda = \text{lk}_1 \Sigma \), \( \Delta = \text{del}_1 \Sigma \),

\[
\prod_{\sigma \in \tilde{\Lambda}} X_{\sigma} \prod_{r \in (\text{del}_1 \Sigma)} \frac{r \sum_{i=1}^{r} X_i}{X_1}.
\]

**Example** bipyramid \( \Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle \) again

\[
\Lambda = \text{lk}_1 \Sigma = \langle 23, 24, 25, 34, 35 \rangle
\]

\[
\Delta = \text{del}_1 \Sigma = \langle 234, 235 \rangle
\]
Weighted enumeration of SST’s in shifted complexes

Theorem Let $\Lambda = \text{lk}_1 \Sigma$, $\Delta = \text{del}_1 \Sigma$.

Example bipyramid $\Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle$ again

$\Lambda = \text{lk}_1 \Sigma = \langle 23, 24, 25, 34, 35 \rangle$

$\Delta = \text{del}_1 \Sigma = \langle 234, 235 \rangle$
Weighted enumeration of SST’s in shifted complexes

**Theorem** Let \( \Lambda = \text{lk}_1 \Sigma \), \( \tilde{\Lambda} = 1 \ast \Lambda \), \( \Delta = \text{del}_1 \Sigma \), \( \tilde{\Delta} = 1 \ast \Delta \).

**Example** bipyramid \( \Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle \) again

\[
\Lambda = \text{lk}_1 \Sigma = \langle 23, 24, 25, 34, 35 \rangle \quad \tilde{\Lambda} = \langle 123, 124, 125, 134, 135 \rangle \\
\Delta = \text{del}_1 \Sigma = \langle 234, 235 \rangle \quad \tilde{\Delta} = \langle 1234, 1235 \rangle
\]
Weighted enumeration of SST’s in shifted complexes

**Theorem** Let $\Lambda = \text{lk}_1 \Sigma$, $\tilde{\Lambda} = 1 \ast \Lambda$, $\Delta = \text{del}_1 \Sigma$, $\tilde{\Delta} = 1 \ast \Delta$.

$$h_d = \prod_{\sigma \in \tilde{\Lambda}} X_{\sigma} \prod r (\sum_{i=1} X_i) / X_1).$$

**Example** bipyramid $\Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle$ again

$$\begin{align*}
\Lambda &= \text{lk}_1 \Sigma = \langle 23, 24, 25, 34, 35 \rangle \\
\tilde{\Lambda} &= \langle 123, 124, 125, 134, 135 \rangle \\
\Delta &= \text{del}_1 \Sigma = \langle 234, 235 \rangle \\
\tilde{\Delta} &= \langle 1234, 1235 \rangle \\
\end{align*}$$

$$h_2 = (123)(124)(125)(134)(135)((1+2+3)/1)((1+2+3+4+5)/1)$$
Cubical complexes

To make boundary work in systematic way, take subcomplexes of high-enough dimensional cube (or, also possible to just define polyhedral boundary map).
Cubical complexes

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- Then we can define boundary map, and all the algebraic topology, including Laplacian.
Cubical complexes

- To make boundary work in systematic way, take subcomplexes of high-enough dimensional cube (or, also possible to just define polyhedral boundary map).
- Then we can define boundary map, and all the algebraic topology, including Laplacian.
- Analogues of Simplicial Matrix Tree Theorems follow readily (in fact for polyhedral complexes).
Complete skeleta (Example)

Spanning trees of 2-skeleton of 4-cube, with appropriate weighting:

\[ p(123)p(124)p(134)p(234)p(1234)^2 \]

where, for instance,

\[ p(123) = x_1 x_2 x_3 y_1 y_2 y_3 \left( \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} \right) \]
Cubical analogue of shifted complexes

- Pick definition of “shifted” to be nice with Laplacians
- In unweighted case, Laplacian eigenvalues are still integers
- Still working on trees