Simplicial spanning trees

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Counting weighted spanning trees of K_n

Theorem [Cayley]: K_n has n^{n-2} spanning trees.

T spanning tree: set of edges containing all vertices and

1. connected ($\tilde{H}_0(T) = 0$)
2. no cycles ($\tilde{H}_1(T) = 0$)
3. $|T| = n - 1$

Note: Any two conditions imply the third.

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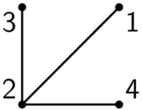
$$\sum_{T \in ST(K_n)} \text{wt } T = (x_1 \cdots x_n)(x_1 + \cdots + x_n)^{n-2}$$

Example: K_4

- 4 trees like: $T =$ 

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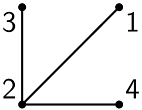
- 4 trees like: $T =$ 

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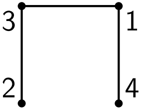
- 12 trees like: $T =$ 

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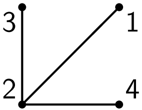
- 4 trees like: $T =$ 

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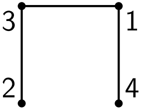
- 12 trees like: $T =$ 

wt $T = (x_1x_2x_3x_4)x_1x_3$

Total is $(x_1x_2x_3x_4)(x_1 + x_2 + x_3 + x_4)^2$.

Laplacian

Definition The Laplacian matrix of graph G, denoted by $L(G)$.

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Definition The reduced Laplacian matrix of graph G, denoted by $L_r(G)$.

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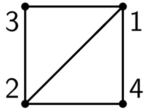
"Reduced": remove rows/columns corresponding to any one vertex

Example



$\partial =$

	12	13	14	23	24
1	-1	-1	-1	0	0
2	1	0	0	-1	-1
3	0	1	0	1	0
4	0	0	1	0	1

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

Matrix-Tree Theorems

Version I Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then G has

$$\frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}$$

spanning trees.

Version II G has $|\det L_r(G)|$ spanning trees

Proof [Version II]

$$\det L_r(G) = \det \partial_r(G) \partial_r(G)^T = \sum_T (\det \partial_r(T))^2$$
$$= \sum_T (\pm 1)^2$$

by Binet-Cauchy

Weighted Matrix-Tree Theorem

$$\sum_{T \in ST(G)} \text{wt}\, T = |\det \hat{L}_r(G)|,$$

where $\hat{L}$ is weighted Laplacian.

Defn 1: $\hat{L}(G) = \hat{D}(G) - \hat{A}(G)$

$\hat{D}(G) = \text{diag}(\hat{\deg} v_1, \ldots, \hat{\deg} v_n)$

$\hat{\deg} v_i = \sum_{v_i v_j \in E} x_i x_j$

$\hat{A}(G)$ = adjacency matrix

(entry $x_i x_j$ for edge $v_i v_j$)

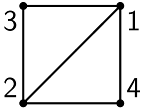
Defn 2: $\hat{L}(G) = \partial(G) B(G) \partial(G)^T$

$\partial(G)$ = incidence matrix

$B(G)$ diagonal, indexed by edges,

entry $\pm x_i x_j$ for edge $v_i v_j$

Example



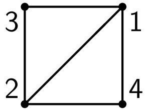
$$\hat{L} = \begin{pmatrix} 1(2+3+4) & -12 & -13 & -14 \\ -12 & 2(1+3+4) & -23 & -24 \\ -13 & -23 & 3(1+2) & 0 \\ -14 & -24 & 0 & 4(1+2) \end{pmatrix}$$

$$\det \hat{L}_r = (1234)(1+2)(1+2+3+4)$$

Threshold graphs

- Vertices $1, \ldots, n$

Example



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- $E \in \mathcal{E}, i \notin E, j \in E, i < j \Rightarrow E \cup i - j \in \mathcal{E}$.

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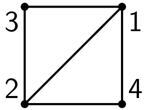

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- Equivalently, the edges form an initial ideal in the componentwise partial order.

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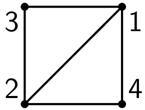
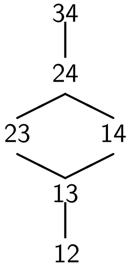


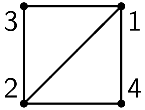


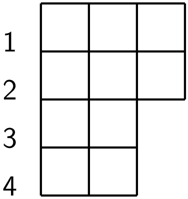
Weighted spanning trees of threshold graphs

Theorem [Martin-Reiner '03; implied by Remmel-Williamson '02]:
If G is threshold, then

$$\sum_{T \in ST(G)} \mathrm{wt}\, T = (x_1 \cdots x_n) \prod_{r \neq 1} \left(\sum_{i=1}^{(d^T)_r} x_i \right).$$

Example





$(1234)(1+2)(1+2+3+4)$

Complete skeleta of simplicial complexes

Simplicial complex $\Sigma \subseteq 2^V$;

$$F \subseteq G \in \Sigma \Rightarrow F \in \Sigma.$$

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Complete skeleton The k-dimensional complete complex on n vertices, *i.e.*,

$$K_n^k = \{F \subseteq V \colon |F| \leq k+1\}$$

(so $K_n = K_n^1$).

Simplicial spanning trees of K_n^k [Kalai, '83]

$\Upsilon \subseteq K_n^k$ is a **simplicial spanning tree** of K_n^k when:

0. $\Upsilon_{(k-1)} = K_n^{k-1}$ ("spanning");
1. $\tilde{H}_{k-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_k(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $|\Upsilon| = \binom{n-1}{k}$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $k = 1$, coincides with usual definition.

Counting simplicial spanning trees of K_n^k

Conjecture [Bolker '76]

$$\sum_{\Upsilon \in SST(K_n^k)} \qquad = n^{\binom{n-2}{k}}$$

Counting simplicial spanning trees of K_n^k

Theorem [Kalai '83]

$$\sum_{\Upsilon \in SST(K_n^k)} |\tilde{H}_{k-1}(\Upsilon)|^2 = n^{\binom{n-2}{k}}$$

Weighted simplicial spanning trees of K_n^k

As before,

$$\mathrm{wt}\,\Upsilon = \prod_{F\in\Upsilon} \mathrm{wt}\,F = \prod_{F\in\Upsilon} \Big(\prod_{v\in F} x_v\Big)$$

Example:

$$\Upsilon = \{123, 124, 125, 134, 135, 245\}$$

$$\mathrm{wt}\,\Upsilon = x_1^5 x_2^4 x_3^3 x_4^3 x_5^3$$

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Theorem [Kalai, '83]

$$\sum_{\Upsilon \in SST(K_n)} |\tilde{H}_{k-1}(\Upsilon)|^2 (\operatorname{wt} \Upsilon) = (x_1 \cdots x_n)^{\binom{n-2}{k-1}} (x_1 + \cdots + x_n)^{\binom{n-2}{k}}$$

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$$\sum_{\Upsilon\in SST(K_n)} |\tilde{H}_{k-1}(\Upsilon)|^2 (\operatorname{wt}\Upsilon) = (x_1\cdots x_n)^{\binom{n-2}{k-1}}(x_1+\cdots+x_n)^{\binom{n-2}{k}}$$

(Adin ('92) did something similar for complete r-partite complexes.)

Proof

Proof uses determinant of reduced Laplacian of K_n^k. "Reduced" now means pick one vertex, and then remove rows/columns corresponding to all $(k-1)$-dimensional faces containing that vertex.

$L = \partial\partial^T$

$\partial \colon \Delta_k \to \Delta_{k-1}$ boundary

$\partial^T \colon \Delta_{k-1} \to \Delta_k$ coboundary

Weighted version: Multiply column F of ∂ by x_F

Example $n = 4, k = 2$

$\partial^T =$

	12	13	14	23	24	34
123	-1	1	0	-1	0	0
124	-1	0	1	0	-1	0
134	0	-1	1	0	0	-1
234	0	0	0	-1	1	-1

$$L = \begin{pmatrix} 2 & -1 & -1 & 1 & 1 & 0 \\ -1 & 2 & -1 & -1 & 0 & 1 \\ -1 & -1 & 2 & 0 & -1 & -1 \\ 1 & -1 & 0 & 2 & -1 & 1 \\ 1 & 0 & -1 & -1 & 2 & -1 \\ 0 & 1 & -1 & 1 & -1 & 2 \end{pmatrix}$$

Simplicial spanning trees of arbitrary simplicial comlexes

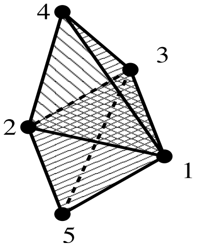
Let Σ be a d-dimensional simplicial complex.
$\Upsilon \subseteq \Sigma$ is a **simplicial spanning tree** of Σ when:

0. $\Upsilon_{(d-1)} = \Sigma_{(d-1)}$ ("spanning");
1. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group ("connected");
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic");
3. $f_d(\Upsilon) = f_d(\Sigma) - \tilde{\beta}_d(\Sigma) + \tilde{\beta}_{d-1}(\Sigma)$ ("count").

- If 0. holds, then any two of 1., 2., 3. together imply the third condition.
- When $d = 1$, coincides with usual definition.

Example

Bipyramid with equator, $\langle 123, 124, 125, 134, 135, 234, 235 \rangle$



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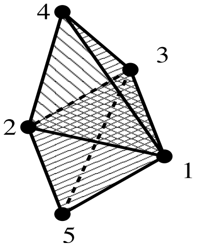

- 6 SST's not containing face 123
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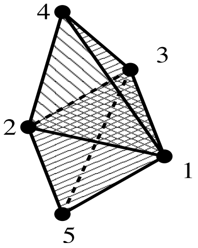

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Total is $(x_1 x_2 x_3)^3 (x_4 x_5)^2 (x_1 + x_2 + x_3)(x_1 + x_2 + x_3 + x_4 + x_5)$.

Simplicial Matrix-Tree Theorem — Version I

- Σ a d-dimensional "metaconnected" simplicial complex
- $(d-1)$-dimensional **(up-down) Laplacian** $L_{d-1} = \partial_{d-1}\partial_{d-1}^T$
- s_d = product of nonzero eigenvalues of L_{d-1}.

Theorem [DKM]

$$h_d := \sum_{\Upsilon \in SST(\Sigma)} |\tilde{H}_{d-1}(\Upsilon)|^2 = \frac{s_d}{h_{d-1}} |\tilde{H}_{d-2}(\Sigma)|^2$$

Simplicial Matrix-Tree Theorem — Version II

- $\Gamma \in SST(\Sigma_{(d-1)})$
- ∂_Γ = restriction of ∂_d to faces not in Γ
- reduced Laplacian $L_\Gamma = \partial_\Gamma \partial_\Gamma^*$

Theorem [DKM]

$$h_d \;=\; \sum_{\Upsilon \in SST(\Sigma)} |\tilde{H}_{d-1}(\Upsilon)|^2 \;=\; \frac{|\tilde{H}_{d-2}(\Sigma;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma;\mathbb{Z})|^2} \det L_\Gamma .$$

Note: The $|\tilde{H}_{d-2}|$ terms are often trivial.

Weighted Simplicial Matrix-Tree Theorems

- Introduce an indeterminate x_F for each face $F \in \Delta$
- Weighted boundary $\boldsymbol{\partial}$: multiply column F of (usual) ∂ by x_F
- $\boldsymbol{\partial}_\Gamma$ = restriction of $\boldsymbol{\partial}_d$ to faces not in Γ
- Weighted reduced Laplacian $\mathbf{L}_\Gamma = \boldsymbol{\partial}_\Gamma \boldsymbol{\partial}_\Gamma^*$

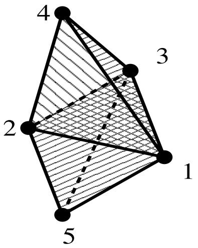
Theorem [DKM]

$$\mathbf{h}_d := \sum_{\Upsilon \in SST(\Sigma)} |\tilde{H}_{d-1}(\Upsilon)|^2 \prod_{F \in \Upsilon} x_F^2 = \frac{\mathbf{s}_d}{\mathbf{h}_{d-1}} |\tilde{H}_{d-2}(\Sigma)|^2$$

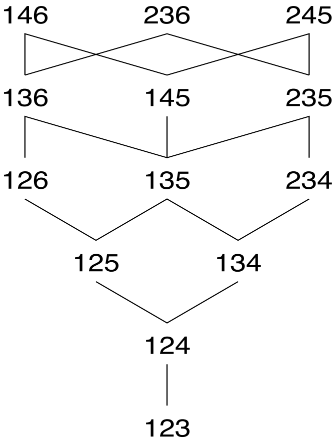
$$\mathbf{h}_d = \frac{|\tilde{H}_{d-2}(\Delta; \mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Gamma; \mathbb{Z})|^2} \det \mathbf{L}_\Gamma.$$

Definition of shifted complexes

- Vertices $1, \ldots, n$
- $F \in \Sigma, i \notin F, j \in F, i < j \Rightarrow F \cup i - j \in \Sigma$
- Equivalently, the k-faces form an initial ideal in the componentwise partial order.
- **Example** (bipyramid with equator)
  $\langle 123, 124, 125, 134, 135, 234, 235 \rangle$



Hasse diagram



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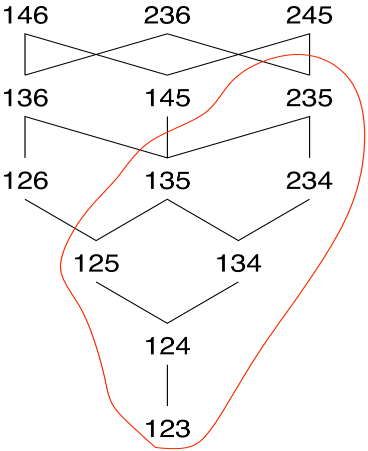

Links and deletions

- Deletion, $\mathrm{del}_1 \Sigma = \{G \colon 1 \notin G, G \in \Sigma\}$.
- Link, $\mathrm{lk}_1 \Sigma = \{F - 1 \colon 1 \in F, F \in \Sigma\}$.
- Deletion and link are each shifted, with vertices $2, \ldots, n$.
- **Example**:

$$\Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle$$

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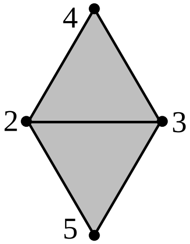
$$\operatorname{del}_1 \Sigma = \langle 234, 235 \rangle$$



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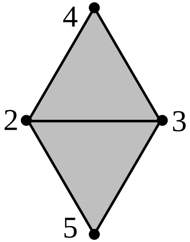
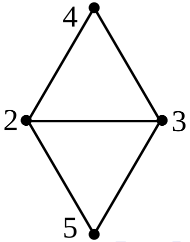
- Deletion and link are each shifted, with vertices $2, \ldots, n$.
- **Example**:

$$\begin{aligned} \Sigma &= \langle 123, 124, 125, 134, 135, 234, 235 \rangle \\ \mathrm{del}_1 \Sigma &= \langle 234, 235 \rangle \\ \mathrm{lk}_1 \Sigma &= \langle 23, 24, 25, 34, 35 \rangle \end{aligned}$$





Weighted enumeration of SST's in shifted complexes

Theorem Let $\Lambda = \operatorname{lk}_1 \Sigma$, , $\Delta = \operatorname{del}_1 \Sigma$,

Example bipyramid $\Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle$ again

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$\Lambda = \text{lk}_1 \Sigma = \langle 23, 24, 25, 34, 35 \rangle$ $\quad \tilde{\Lambda} = \langle 123, 124, 125, 134, 135 \rangle$

$\Delta = \text{del}_1 \Sigma = \langle 234, 235 \rangle$ $\quad \tilde{\Delta} = \langle 1234, 1235 \rangle$

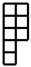
Weighted enumeration of SST's in shifted complexes

Theorem Let $\Lambda = \mathrm{lk}_1 \Sigma$, $\tilde{\Lambda} = 1 * \Lambda$, $\Delta = \mathrm{del}_1 \Sigma$, $\tilde{\Delta} = 1 * \Delta$.

$$\mathbf{h}_d = \prod_{\sigma \in \tilde{\Lambda}} X_\sigma \prod_r \Big(\Big(\sum_{i=1}^{(d(\tilde{\Delta})^T)_r} X_i\Big)/X_1\Big).$$

Example bipyramid $\Sigma = \langle 123, 124, 125, 134, 135, 234, 235 \rangle$ again

$\Lambda = \mathrm{lk}_1 \Sigma = \langle 23, 24, 25, 34, 35 \rangle$ $\tilde{\Lambda} = \langle 123, 124, 125, 134, 135 \rangle$

$\Delta = \mathrm{del}_1 \Sigma = \langle 234, 235 \rangle$ $\tilde{\Delta} = \langle 1234, 1235 \rangle$ 

$$\mathbf{h}_2 = (123)(124)(125)(134)(135)((1+2+3)/1)((1+2+3+4+5)/1)$$

Cubical complexes

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- Analogues of Simplicial Matrix Tree Theorems follow readily (in fact for polyhedral complexes).

Complete skeleta (Example)

Spanning trees of 2-skeleton of 4-cube, with appropriate weighting:

$$p(123)p(124)p(134)p(234)p(1234)^2$$

where, for instance,

$$p(123) = x_1x_2x_3y_1y_2y_3(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3})$$

Cubical analogue of shifted complexes

- Pick definition of "shifted" to be nice with Laplacians
- In unweighted case, Laplacian eigenvalues are still integers
- Still working on trees