Matroids and statistical dependency

Art Duval, Amy Wagler

University of Texas at El Paso

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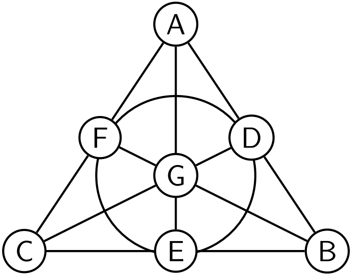
- Yes. For instance, $Z = 1 + XY + \epsilon$.
- We might expect to get any sort of simplicial complex (subsets of independent sets are independent).
- We can even get the Fano plane: A, B, C independent, $D = AB, E = BC, F = CA, G = DEF$.



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- In regression modeling, matroid structures could be used as a variable selection procedure to find the most parsimonious set of X's to predict a Y. The results of the minimally dependent sets [circuits] would also inform which interactions ($x_1 x_2$ products) should be investigated for inclusion to the model.
- In big data settings, a matroid would identify maximally independent sets [bases] so that multiplicity can be corrected at the circuit level rather than the full data set.

How to picture data

Each variable is a vector, whose components are measurements of this variable.

- m different variables
- n different trials
- m vectors in $\mathbb{R}^n$

Example

Three variables, four trials

$$\begin{array}{l} X = (3.1 \quad 1 \quad 4 \quad 2\ ) \\ Y = (\ 2 \quad 1 \quad 6.9 \quad 8\ ) \\ Z = (\ 5 \quad 2.1 \quad 11 \quad 9.9) \end{array}$$

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Question

How can we identify statistically dependent sets in general? And capture non-linear dependence? What is "close enough"?

Joint cumulants

Definition

$$\prod_{a=1}^{b(\tau)} E(\prod_{i\in\tau_a} X_i) = \sum_{\sigma\leq\tau} \kappa_\sigma$$

By Möbius inversion, we can solve for κ's.

Example

$$E(X_1)E(X_2)E(X_3)E(X_4) = \kappa_{1|2|3|4}$$

$$E(X_1X_2)E(X_3)E(X_4) = \kappa_{1|2|3|4} + \kappa_{12|3|4}$$

So $\kappa_{12|3|4} = (E(X_1X_2) - E(X_1)E(X_2))E(X_3)E(X_4)$

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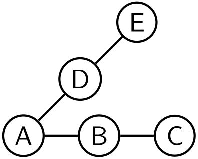
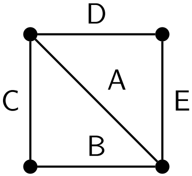
- And cumulants behave nicely enough to rigorously test statistical significance of distance from zero on actual data.
    - Cumulants are U-statistics and asymptotically normally distributed.
    - Cumulants have easier interpretive value.

Matroids

Matroids make abstract ideas of independence, and model

- linear independence and dependence of sets of vectors in linear algebra;
- independent (cycle-free) sets of edges in graphs;
- etc.





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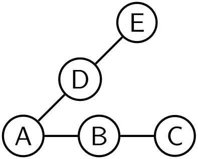

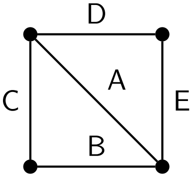

Remark

Not all matroids can be represented by vectors or graphs

Transitivity

Not all data can be represented by matroids.

Matroids: If $\{x, y\}$ are dependent and $\{y, z\}$ are dependent, then $\{x, z\}$ are dependent.

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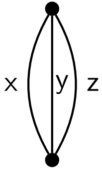

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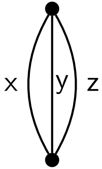
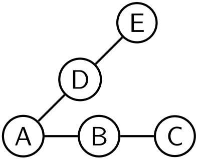
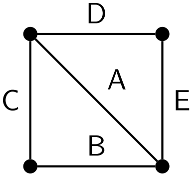


Statistics: Not always! But we will look for conditions on data that allow dependence to be modeled by matroids.

Independent sets

- $\emptyset$ is independent.
- Any subset of an independent set is also independent.
- If I_1, I_2 independent, and $|I_2| = |I_1| + 1$, then $\exists x \in I_2 - I_1$ such that $I_1 \cup \{x\}$ is independent.

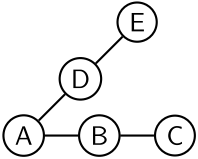


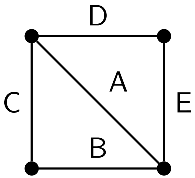


Bases

Maximally independent sets

- $\emptyset$ is not a basis.
- One basis cannot be a proper subset of another basis.
- If B_1, B_2 are bases and $x \in B$, then $\exists y \in B_2$ such that $(B_1 - \{x\}) \cup \{y\}$ is a basis.

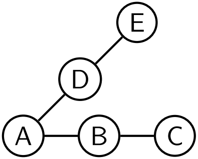


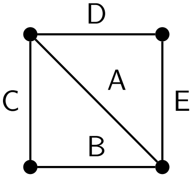


Circuits

Minimally dependent sets

- $\emptyset$ is not a circuit.
- One circuit cannot be a proper subset of another circuit.
- $(C_1 \cup C_2) - \{x\}$ contains a circuit for distinct circuits C_1, C_2.

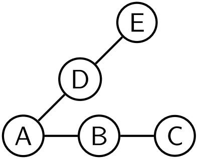


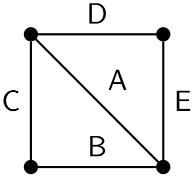


Rank function

Size of maximal independent subset of a set

- $r(\emptyset) = 0$.
- $r(S \cup \{x\}) = r(S)$ or $r(S) + 1$.
- If $r(S) = r(S \cup \{x\}) = r(S \cup \{y\})$, then $r(S \cup \{x, y\}) = r(S)$.





Closure axioms

A matroid on ground set E may be defined by closure axioms:

$$\mathrm{cl}\colon 2^E \to 2^E$$

- Closure axioms:
  - $A \subseteq \mathrm{cl}(A)$
  - If $A \subseteq B$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$
  - $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$
- Exchange axiom: If $x \in \mathrm{cl}(A \cup y) - \mathrm{cl}(A)$, then $y \in \mathrm{cl}(A \cup x)$

For us, $x \in \mathrm{cl}(A)$ means that knowing the values of all the variables in A implies knowing something about the value of x. (Sort of: x is a function of A, with statistical noise and fuzziness.)

Invertibility

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- $x \in \mathrm{cl}(A \cup y) - \mathrm{cl}(A)$ means that in using $A \cup y$ to determine x, we must use (can't ignore) y. ("model parsimony")
- $y \in \mathrm{cl}(A \cup x)$ means we can "solve" for y in terms of x and A. (This is sort of invertibility.)

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Easiest way for a function (only way for continuous function) to be invertible is to be monotone in each variable. Fortunately, implied by a common statistical assumption:

Definition (MTP$_2$)

(Multivariate Totally Positive of order 2.)
$f(u)f(v) \leq f(u \wedge v)f(u \vee v)$, where f is probability distribution, u and v are vectors of variable values, and $\wedge$ and $\vee$ denote element-wise minimum and maximum.

Multivariate Totally Positive of order 2

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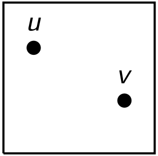

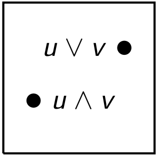

Composition

Closure axioms

- $A \subseteq \mathrm{cl}(A)$ (easy)
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Example

When $A = x$ is a single element and $\mathrm{cl}(x) = \{x, y\}$. We need to avoid $z \in \mathrm{cl}\{x, y\}$ for $z \neq x, y$. In other words, z depends on y, and y depends on x should mean that z depends on x directly. This is a kind of transitivity.

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More generally, if Z is determined by $Y_1, \ldots, Y_p$, and each Y_i is determined by $X_1, \ldots, X_q$, then Z should be determined directly by $X_1, \ldots, X_q$. This is a kind of composition.

Remark

MTP_2 means the dependence will be strong enough to guarantee transitivity, and more generally composition.

Dependence axioms

How we actually show that we have a matroid. The dependent sets $\mathcal{D}$ in a matroid satisfy:

1. $\emptyset \notin \mathcal{D}$
2. If $D \in \mathcal{D}$ and $D' \supseteq D$, then $D' \in \mathcal{D}$
3. If $I \notin \mathcal{D}$, but $I \cup \{x, y\}, I \cup \{y, z\} \in \mathcal{D}$, then $I \cup \{x, z\} \in \mathcal{D}$.

We can prove that MTP_2 distributions satisfy this, using singleton-transitivity of *conditional dependence* when data is MTP_2.

Example: Cancer genes

Non-matroid analysis: Clusters

$$\{1, 3, 4\},\ \{2, 5, 6, 7, 13\},\ \{8, 9, 11, 12\},\ \{10\}.$$

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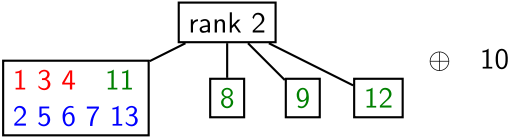

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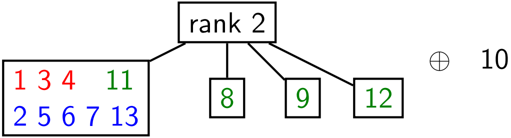

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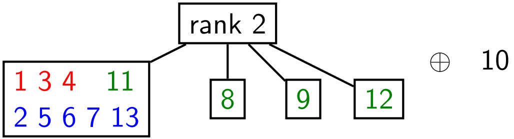

Remark

This suggests two independent, possibly latent, variables explaining the left side of the diagram.