Section 9.2

Infinite Series
If \( \{a_n\} \) is an infinite sequence, then
\[
\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots
\]
is an infinite series.

Definitions of Convergent and Divergent Series
For the infinite series \( \sum_{n=1}^{\infty} a_n \), the \( n \)th partial sum is given by
\[
S_n = a_1 + a_2 + \ldots + a_n.
\]
If the sequence of partial sums \( \{S_n\} \) converges to \( S \), then the series \( \sum_{i=1}^{\infty} a_n \) converges. The limit \( S \) is called the sum of the series.
\[
S = a_1 + a_2 + \ldots + a_n + \ldots
\]
If \( \{S_n\} \) diverges, then the series diverges.

Telescoping Series
\[
(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \ldots
\]

Geometric Series
The series given by
\[
\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots + ar^n + \ldots, \quad a \neq 0
\]
is a geometric series with ratio \( r \).

Convergence of a Geometric Series
A geometric series with ratio \( r \) diverges if \( |r| \geq 1 \). If \( 0 < |r| < 1 \), then the series converges to the sum
\[
\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}, \quad 0 < |r| < 1.
\]

Properties of Infinite Series
If \( \sum a_n = A \), \( \sum b_n = B \), and \( c \) is a real number, then the following series converge to the indicated sums,

a) \( \sum_{n=1}^{\infty} ca_n = cA \)

b) \( \sum_{n=1}^{\infty} (a_n + b_n) = A + B \)

c) \( \sum_{n=1}^{\infty} (a_n - b_n) = A - B \)
Limit of $n$th term of a Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.

$n$th-Term Test for Divergence

If $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Problem 1. Verify that the infinite series diverges.

a) $\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n$

b) $\sum_{n=1}^{\infty} \frac{n}{2n + 3}$

Problem 2. Verify that the infinite series converges, and find its sum.

a) $\sum_{n=1}^{\infty} \frac{1}{n(n + 2)}$
b) \( \sum_{n=1}^{\infty} 2 \left( -\frac{1}{2} \right)^n \)

c) \( \sum_{n=1}^{\infty} \frac{1}{n^2 - 1} \)

d) \( 4 + 6 + \frac{9}{2} + \frac{27}{8} + \ldots \)
Problem 3. Write the repeating decimal 0.0\overline{1} as a geometric series, and write its sum as the ratio of two integers.

Problem 4. Determine the convergence or divergence of the series.

a) \[ \sum_{n=1}^{\infty} \frac{n + 1}{2n - 1} \]

b) \[ \sum_{n=1}^{\infty} \frac{3^n}{n^3} \]

c) \[ \sum_{n=1}^{\infty} \ln \left( \frac{n + 1}{n} \right) \]