

Exploring Machin's Approximation of π

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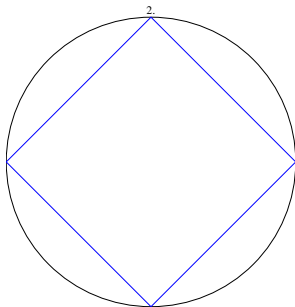
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Archimedes of Syracuse (≈ 287 – 212 BC) approximated π by the Method of Exhaustion:



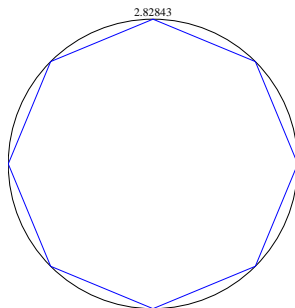
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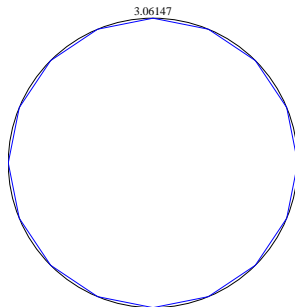
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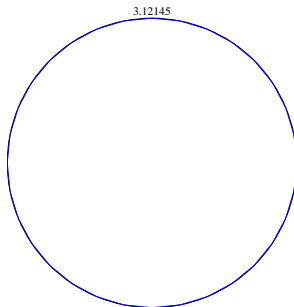
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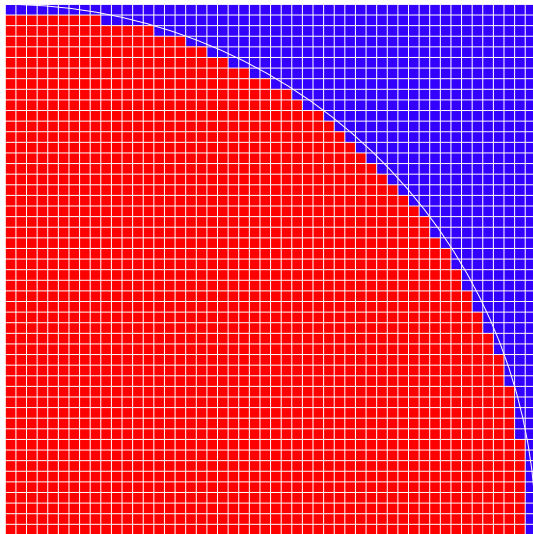
The accuracy of the approximation increases by about 1 digit for every 2 iterations.

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3.0576



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James Gregory (1638–1675) found the McLaurin series expansion for the arctangent function:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \dots$$

The radius of convergence of the series is 1.

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As a consequence we obtain **Leibniz' Formula**:

$$\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

(The series converges by the Alternating Series Test.)

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(The series converges by the Alternating Series Test.)

Convergence of this series is very slow:

Summing 10,000 terms of the series yields 3.14149 as an approximation of π .

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This is a nice illustration of the **General Calculus Philosophy**:

Convergence behavior is good, if x is close to the center of a Taylor series; convergence is bad if x is close to the “edge” of convergence.

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Therefore **John Machin** (≈ 1686 -1751) [and others] had the following idea:

Replace the 1 in Leibniz' Formula by one or more numbers closer to 0.

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The ingredient needed is the **addition formula** for the tangent function:

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

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Applying the addition formula twice we obtain:

$$\tan(4\alpha) = \frac{4 \tan(\alpha) - 4 \tan^3(\alpha)}{1 + \tan^4(\alpha) - 6 \tan^2(\alpha)}$$

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Applying the addition formula twice we obtain:


$$\tan(4\alpha) = \frac{4 \tan(\alpha) - 4 \tan^3(\alpha)}{1 + \tan^4(\alpha) - 6 \tan^2(\alpha)}$$

If we set $x = \tan \alpha$ and $y = \tan \beta$, we can apply the addition formula one more time to get

$$\tan(4\alpha + \beta) = \frac{x^4 y - 4x^3 - 6x^2 y + 4x + y}{x^4 + 4x^3 y - 6x^2 - 4xy + 1}$$

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Using this for

$$\frac{\pi}{4} = 4\alpha + \beta,$$

we get the equation:

$$1 = \frac{x^4 y - 4x^3 - 6x^2 y + 4x + y}{x^4 + 4x^3 y - 6x^2 - 4xy + 1}$$

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At this point, Machin made the decision to choose $x = \frac{1}{5}$, then solved for y to obtain $y = -\frac{1}{239}$.

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Since the angles $\alpha = \arctan\left(\frac{1}{5}\right)$ and $\beta = \arctan\left(-\frac{1}{239}\right)$ satisfy the equation

$$\frac{\pi}{4} = 4\alpha + \beta,$$

we finally arrive at Machin's Formula

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$$

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Machin then used this formula in 1706 to compute 100 digits of π .

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Using similar ideas and identities such as

$$\frac{\pi}{4} = 44 \arctan \frac{1}{57} + 7 \arctan \frac{1}{239} - 12 \arctan \frac{1}{682} \\ + 24 \arctan \frac{1}{12943},$$

found by **Carl Størmer** in 1896,

Yasumasa Kanada (1949–) and his collaborators computed more than 1.24 trillion digits of π in 2002.

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References:

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