

A Capstone Course for Future High School Mathematics Teachers

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Overview

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- 2 Course Structure
- 3 Wu's Principles
- 4 References

Basic Characteristics Important for K–12 Math Teaching

by H. Wu

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- **Precision:** Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known.

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- **Precision:** Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known.
- **Definitions:** They are the bedrock of the mathematical structure. They are the platform that supports reasoning. No definitions, no mathematics.
- **Reasoning:** The lifeblood of mathematics. The engine that drives problem solving. Its absence is the root cause of teaching—and learning—by rote.

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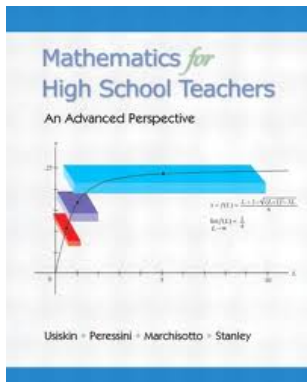
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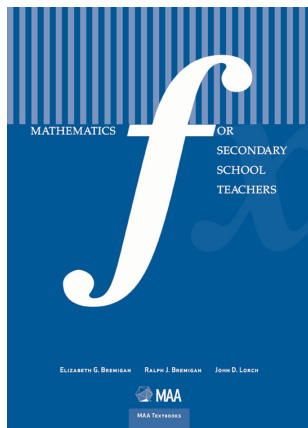
- **Coherence:** Mathematics is a tapestry in which all the concepts and skills are interwoven. It is all of a piece.
- **Purposefulness:** Mathematics is goal-oriented, and every concept or skill is there for a purpose. Mathematics is not just fun and games.

Common Core State Standards: Mathematical Practices (for Pupils)

- 1 Make sense of problems and persevere in solving them.
- 2 Reason abstractly and quantitatively.
- 3 Construct viable arguments and critique the reasoning of others.
- 4 Model with mathematics.
- 5 Use appropriate tools strategically.
- 6 Attend to precision.
- 7 Look for and make use of structure.
- 8 Look for and express regularity in repeated reasoning.

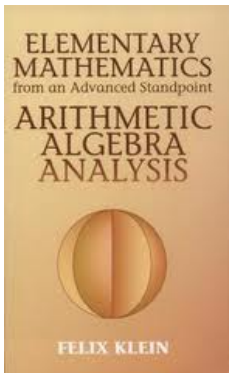


Zalman Usiskin, Anthony L. Peressini,
Elena Marchisotto, and Dick Stanley:
Mathematics for High School Teachers.



Elizabeth G. Bremigan, Ralph J. Bremigan, and John D. Lorch:
Mathematics for Secondary School Teachers

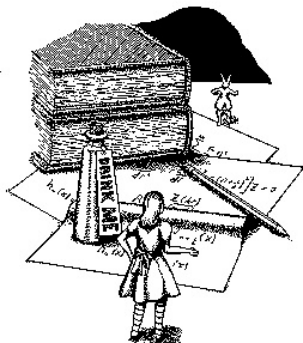




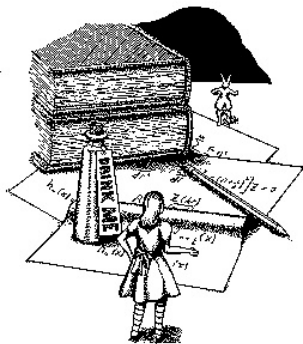
Felix Klein:

Elementary Mathematics from an Advanced Standpoint
Arithmetic, Algebra, Analysis

First published in 1908

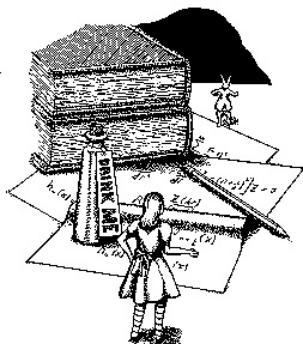


Course Content:



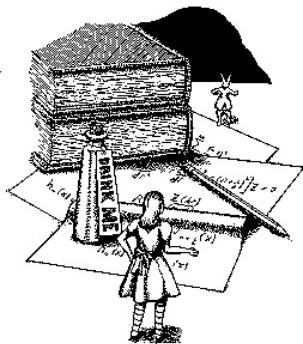
Course Content:

- From the Natural Numbers to the Real Numbers



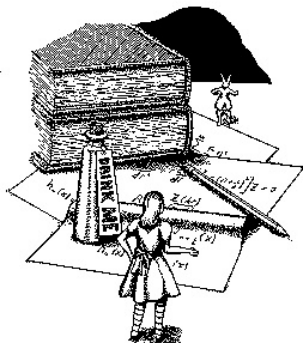
Course Content:

- From the Natural Numbers to the Real Numbers
- Complex Numbers



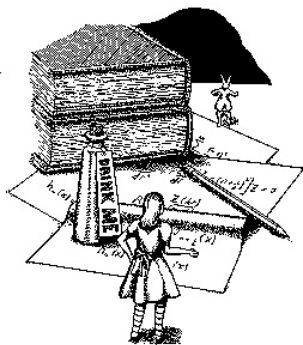
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Prerequisite:

- Introduction to Logic and Proof

Wu's Principles and Student Beliefs: 1. Precision

- Students make progress during the course in understanding that they need to transition from being “Math learners” to “Math talkers”.

Student Group Activity I

- Teach a Lesson

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- Teach a Lesson
 - Each student group teaches 1–2 sections in the Equations chapter
 - Fellow classmates present homework solutions - lesson presenters critique the solutions.
 - Duration: 50–80 minutes
 - Each group needs to complete a trial run with the instructor before teaching the lesson

Student Group Activity II

- Final Paper & Presentation

Student Group Activity II

- Final Paper & Presentation
 - Countability of algebraic numbers
 - Algebraic numbers form a field
 - Stereographic projection
 - Transcendence of the number e (à la Felix Klein)
 - Cardano-Tartaglia method for solving cubic equations (incl. *Casus Irreducibilis*)
 - Newton's Method

Wu's Principles and Student Beliefs: 2. Definitions

- Students do not understand the role of definitions in Mathematics and/or believe that definitions are not important for HS Mathematics.
- Students are not good as “digesting” definitions.

Example: Class Activity

- Define “chair”.
- Define “prime number”.

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- Define “chair”.
- Define “prime number”.

Subsequent discussion of the differences between the two definitions. . .

Students say that mathematical definitions are more precise than other definitions; the words “descriptive vs. normative” do not surface.

Example: Group Activity

- Recall that a *relation on a set* X is a subset of the Cartesian product $X \times X$.

We define a relation \sim on $\mathbb{N} \times \mathbb{N}$ as follows:

$$(p, q) \sim (p', q') \Leftrightarrow p + q' = p' + q.$$

Show that \sim defines an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

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Show that \sim defines an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

No student group notices that $X = \mathbb{N} \times \mathbb{N}$...

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- Based on their own HS experience(?), students will not expect mathematical reasoning from their pupils.
- Students do not understand that reasoning with definitions will be the only way for them to “check” the Mathematics they will be teaching.

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(after discussing and practicing the Division Algorithm)

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Last semester was the first time that the instructor prompt “How do you compute the decimal expansion of a rational number?” enabled a couple of students to come up with a line of reasoning. . .

Example: Discussion of a Test Problem
(after finishing the Equations chapter)

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Why is $x = 1$ a solution, while $x = 1/4$ isn't?

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- $\Leftrightarrow \frac{x}{1 + \sqrt{x}} = \frac{1}{2}$

- $\Leftrightarrow 2x - 1 = \sqrt{x}$

- $\Rightarrow 4x^2 - 4x + 1 = x$

- $\Leftrightarrow (4x - 1)(x - 1) = 0$

Why is $x = 1$ a solution, while $x = 1/4$ isn't?

Wu's Principles and Student Beliefs: 4. Coherence

- Students do not seem to have been exposed much to the idea that “Mathematics is a tapestry in which all the concepts and skills are interwoven”.
- Mathematics seems to have been taught to them as a collection of more or less unrelated mathematical subjects.

Example: Algebra

- Being essentially an Algebra course, the course probably contributes to the problem.
- The connection between “Modern Algebra” and “HS Algebra” is stressed.

Wu's Principles and Student Beliefs: 5. Purposefulness

- Intrinsic motivation seems to be a completely new concept to students.
- Their HS and college experience for the most part seem to have been mathematically “purposeless”.

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- What are real numbers good for? Where/when in the HS curriculum do we need real numbers?
- What were complex numbers invented for?
- Why are surjectivity and injectivity important properties of functions?

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- What is the purpose of the material you present?
- Does the material you present tell a story?

Example: Questions During the Trial Run

- What is the purpose of the material you present?
- Does the material you present tell a story?
- Why do you present this example? What purpose does it serve?

References

- 1 Matthew Winsor, *One Model of a Capstone Course for Preservice High School Mathematics Teachers*. Primus 19 (2009), pp. 510–518.
- 2 Hung-Hsi Wu, *The Mathematics K–12 Teachers Need to Know*.
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- 3 Hung-Hsi Wu, *The Mis-Education of Mathematics Teachers*. Notices of the AMS 58 (2011).
- 4 Jo Boaler, *What's Math Got to Do with It? How Parents and Teachers Can Help Children Learn Their Least Favorite Subject*. Penguin 2008.