Cyclic Quadrilaterals

Cyclic Quadrilateral
A quadrilateral inscribed on a circle, a quadrilateral whose vertices all lie on a circle. The vertices are said to be concyclic.

Theorem
A necessary condition and sufficient condition for a quadrilateral to be cyclic is that the sum of its opposite angles is 180°.

Theorem
If $\alpha = \beta$ then the quadrilateral $ABCD$ is cyclic.
Problem 1. The circles $C_1$ and $C_2$ intersects at the points $A$ and $B$. A line through the point $A$ intersects the circles $C_1$ and $C_2$ at the points $C$ and $D$, respectively. The tangent line through $C$ to $C_1$ and the tangent line through $D$ to $C_2$ intersects at the point $M$. Show that the quadrilateral $MCBD$ is cyclic.

Problem 2. Let $BC$ the diameter of a semicircle, and let $A$ the midpoint of the semicircle (boundary), $M$ a point over the arc $AC$, and $P, Q$ the foot of the altitudes from $A$ and $C$ to the line $BM$, respectively. Show that $BP = PQ + QC$. 
**Problem 3.** The angle bisectors of the interior angles of a quadrilateral $ABCD$ intersects at the points $E$, $F$, $G$ and $H$, as shown below. Show that the quadrilateral $EFGH$ is cyclic.

**Problem 4.** In a triangle $ABC$, a parallel line to the side $BC$ intersects the side $AB$ and $AC$ at the points $P$ and $Q$, respectively. The circle passing through $P$ and tangent to $AC$ at the point $Q$ intersects $AB$ at $R$. Show that the quadrilateral $RQCB$ is cyclic.
Problem 5. In a rectangle $ABCD$, let $P$ a point in the interior of $ABCD$ such that $\angle APD + \angle BPC = 180^\circ$. Find $\angle DAP + \angle BCP$.

Problem 6. In a triangle $ABC$, let $D$ the foot of the perpendicular from $A$, $E$ and $F$ over a line passing through $D$ such that $AE$ is perpendicular to $BE$, $AF$ is perpendicular to $CF$, with $D \neq E \neq F$. Let $M$ and $N$ the midpoints of $BC$ and $EF$, respectively. Show that $AN$ is perpendicular to $NM$. 
Exercises

Problem 7. Let $ABCD$ a convex quadrilateral which its diagonals are perpendicular, over every side of this quadrilateral a square is constructed exterior to the quadrilateral. Show that the segment joining the centers of the opposite squares (squares over the opposite sides of the quadrilateral) pass through the intersection point of the diagonals of the quadrilateral.

Problem 8. Let $ABCD$ a square, $M$ the midpoint of $AB$. A perpendicular line to $MC$ passing through $M$ intersects $AD$ at $K$. Show that $\angle BCM = \angle KCM$.

Problem 9. Let $ABCD$ a cyclic quadrilateral, let $M$ the intersection point of the diagonals of $ABCD$, and let $E, F, G$ and $H$ the foot of the perpendiculars from $M$ to the sides $AB, BC, CD,$ and $DA$, respectively. Show that $EFGH$ has an inscribed circumference (a circumference tangent to every side of the quadrilateral), and find its center.

Problem 10. Let $AB$ the diameter of a circle with center at $O$. Let $C$ a point over the circumference such that $OC$ is perpendicular to $AB$. Let $P$ a point over the arc $CB$. The lines $CP$ and $AB$ intersects at $Q$. Let a point $R$ over the line $AP$ such that $RQ$ is perpendicular to $AB$. Show that $BQ = QR$.

Problem 11. Let $ABCD$ a cyclic quadrilateral with the property that its diagonals are perpendicular, $P$ the intersection point of its diagonals. Show that the perpendicular line from $P$ to any of the quadrilateral’s sides bisect the opposite side.

Problem 12. Let $ABCD$ a cyclic quadrilateral with the property that its diagonals are perpendicular (intersection angle is $90^\circ$). Show that the distance from the center of the circumcircle of $ABCD$ to one of the sides is the half of the length of the opposite side.

Problem 13. Given is the circle $C_1$, let $P$ an exterior point to $C_1$, from $P$ draw two tangent lines to $C_1$ which intersects $C_1$ at $A$ and $B$, also from $P$ draw a secant line $l$ to $C_1$. From the center $O$ of $C_1$ draw a perpendicular line to $l$, this line intersects $C_1$ at the point $K$, and let $C$ the intersection between $OK$ and $l$. Show that $BK$ is the angle bisector of $\angle ABC$. 