Banach space geometry and the xed point property

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Abstract

This is the text of four lectures given at the special workshop on metric fixed point theory held at the University of Seville from September 25 { 29, 1995. I would like to thank the University of Seville and the Workshop organizers for the invitation to participate in the workshop and for their e orts to stage such a rewarding and scienti cally successful meeting.

1. Foreplay

Throughout $X \equiv (X, \|\cdot\|)$ will denote a real Banach space (although our results remain valid if X over the complex eld). The closed unit ball of X is $B_X := \{x \in X : \|x\| \leq 1\}$, and the unit sphere of X is $S_X := \{x \in X : \|x\| = 1\}$. The dual space of X is the space of all continuous linear functionals $f : X \to \mathbb{R}$ equipped with the norm $\|f\| := \sup f(B_X)$. For each $x \in X$ we de ne the evaluation functional $\hat{x} : X^* \to \mathbb{R} : f \mapsto \hat{x}(f) := f(x)$. The mapping $J : x \mapsto \hat{x}$ is an isometric isomorphism of X into X^{**} and we refer to $\hat{X} := J(X)$ as the natural embedding of X in X^{**}. The space X is re exive if $\hat{X} = X^{**}$. By duality map on X we understand $D : X \to 2^{X^*} : x \mapsto \{f \in X^* : f(x) = \|f\|\|x\| = \|x\|^2\}$. $D(x)$ is nonempty for each $x \in X$ courtesy of the Hahn-Banach theorem, and it is a fundamental result of R. C. James th is re exive if and only if D is onto.

Besides the norm topology we will be concerned with the weak topology on X, $w := \sigma(X, X^*)$, and the weak* topology on X^*, $w^* := \sigma(X^*, X)$. For what follows it will be enough to recall the following.

In the case of the weak topology:

For a sequence (x_n) we have $x_n \xrightarrow{w} x$ if and only if $f(x_n) \to f(x)$, for all $f \in X^*$.

A subset A of X is w-compact if and only if it is w-sequentially compact (the Eberline-Smullian th

The norm function on X is w-lower semi-continuous.

The unit ball B_X is w-compact if and only if X is re exive.

Of 'historical' importance for metric xed point theory is Mazur's theorem for $A \subseteq X$ we have $\overline{\text{co}}\, A = \overline{\text{co}}^w A$.

In the case of the weak* topology:

For a net (f_α) we have $f_\alpha \xrightarrow{w^*} f$ if and only if $f_\alpha(x) \to f(x)$, for all $x \in X$.

w^*-compact sets are w^*-sequentially compact if for example X is separable (in which case the w^*-topology on B_{X^*} is a metric topology), or more generally if X is smoothable (that is, fo equivalent norm on X the duality map D is single valued).

An equivalent norm on X^* is the dual norm of an equivalent norm on X if and only if it is w^*-lower semi-continuous.

$B_{\hat{X}}$ is w^*-dense in B_X (Goldstein's theorem).

We will concentrate on the w and w^* topologies. While many of the results extend to other lo convex linear topologies τ for which the norm is τ-lower semi-continuous, the results and arguments for w or w^* cases are typical.

In all that follows C will denote a nonempty closed bounded convex subset of X, and T a nonexpansive self mapping of C: $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$.

Examples of nonexpansive maps on appropriate domains include the following,

Isometries,

Shifts,

Multiplication operators,

Resolvents of accretive operators,

Contractions (that is, $x \neq y \Rightarrow \|Tx - Ty\| < \|x - y\|$). It is interesting to note that while being a contraction is clearly more restrictive than being a nonexpansive map, apart from the observa contractions can have at most one fixed point, almost nothing more seems to be known for contrac than for nonexpansive maps.

Strict contractions (that is, there exists $k \in [0, 1)$ such that $\|Tx - Ty\| \leq k\|x - y\|$, for all $x, y \in C$). These are the subject of the celebrated Banach contraction mapping principle, which ensu existence of a (necessarily unique) fixed point to which any orbit $x, Tx, T^2x, T^3x, \cdots$ converges in norm convergent at a geometric rate.

We say X has the fixed point property, fpp, if on each nonempty closed bounded convex subset C ev nonexpansive self mapping T has a fixed point.

We say X has the weak fixed point property, w-fpp, if on each nonempty w-compact convex subset every nonexpansive self mapping T has a fixed point.

We say the dual of X, X^*, has the weak* fixed point property, w^*-fpp, if on each nonempty w^*-compa convex subset C every nonexpansive self mapping T has a fixed point.

We will be concerned with sufficient conditions for each of these properties. Of course all three coincide when X is reflexive.

Not all spaces enjoy these properties. For example:

- (1) c_0 fails the fpp. $T(x_1, x_2, x_3, \cdots) \mapsto (1, x_1, x_2, \cdots)$ is a fixed point free isometry on $B_{c_0}^+ = \{(x_n) : x_n \to 0,\ 0 \leq x_n \leq 1\}$.
- (2) $L_1[0, 1]$ fails the w-fpp. The baker transform followed by $f(t) \mapsto 1 - f(t)$ is a fixed point free isometry on the w-compact order interval $0 \leq f(t) \leq 1$. This is a modification by R. [Si] of the celebrated example of D. Alspach [A] which provided the first instance of a Banach failing the w-fpp and so settled a question of more than 20 years standing. The example is ca

in abstract L-spaces [B-S], and remains e ectively the only explicitly known example for the the w{fpp (for an alternative construction see Schechtmann [Sc]).

" (3a) $\ell_1 = c_0^*$ equipped with the equivalent dual norm $\|(x_n)\| := \|(x_n^+)\|_1 \vee \|(x_n)^-\|_1$ fails the w^*{fpp for $T(x_n) := (1 - \sum x_n, x_1, x_2, \cdots)$ acting on the w^*-compact convex set $C := B_{\ell_1}^+$ [Lim].

" (3b)" ℓ_1 as c^* fails the w^*{fpp for an a ne contraction [D-L, Sm]. To see this, it is conveni consider ℓ_1 acting on c via the action $(x_n)(c_n) := x_1 c_1 + x_2 c_2 + x_3 c_3 + \cdots$, for $(x_n) \in \ell_1$ and all $(c_n) \in c$. Then, for any $\delta \in (0,1)$ and sequence $(c_n) \subset (0,1)$ with $\sum \epsilon_n < \infty$ take

$$T(x_n) := \left(\delta(1-x_1) + \sum_{k=1}^{\infty}(1-\epsilon_k)x_{k+1},\ \delta(1-x_1),\ (1-\epsilon_1)x_2,\ (1-\epsilon_2)x_3\right), \cdots$$

acting on the w^*-compact convex subset

$$C := \left\{ (x_n) : 0 \le x_n,\ x_1 = \sum_{k=2}^{\infty} x_k \le 1 \right\}.$$

It is unknown whether or not every Banach space has the w{fpp for contractions.

Henceforth we concentrate on su cient conditions for the w{fpp and w^*{fpp , and so for the fpp case of re exive spaces. It is a signi cant open question as to whether or not the fpp implies r although there is considerable evidence in support of the conjecture that it does. The converse perhaps the fundamental open question in metric xed point theory.

It is an interesting, but seldom used, result that the w-fpp is separably determined. To see this, suppose that C is a nonempty w-compact convex subset and that T is a xed point free nonexpansive mapping o into C. Choose $c \in C$ and define $K_1 := \{c\}$ and inductively $K_{n+1} := \overline{\mathrm{co}}\{T(K_n) \cup K_n\}$. Then

$$K_\infty := \overline{\bigcup_n K_n}$$

is a separable, convex w-compact (by Mazur's theorem) T-invariant subset, K_∞ is an xed point free nonexpansive self mapping.

The above argument actually establishes the formally stronger result that any compact minimal w convex T-invariant set is separable.

A similar result in the w^*-case is potentially important, as then w^*-compactness could be a sequential. Unfortunately no such general result seems known. The situation in $C(\ )^*$ has been inve by M. Smyth [Sm]. If is an in nite compact Hausdor space $C(\ )^*$ failure of the w^*{fpp is separ determined (either it contains an isometric copy of L_1, or one of the Alspach example, or it is an $\ell_1(\ )$ and failure of the w^*{fpp can be reduced to a separable example similar to the ℓ_1 example considered ab however, is not dispersed then $C(\ )^*$ contains a nonseparable w^*-compact convex subset C which a a nonexpansive mapping $T : C \to C$ for which C is a minimal invariant subset. That is, $C(\ )^*$ contains inseparable minimal invariant subsets.

2. Opening moves

Suppose X (X^*) fails the w (w^*)-fpp, then there exists a nonempty w (w^*)-compact convex sub C and a xed point free nonexpansive mapping $T : C \to C$. By the compactness and Zorn's lemma we

may assume that C is minimal in the sense that no proper nonempty w (w^*)-compact convex subset of C is invariant under T. We will refer to such a set as a *minimal invariant set* for T (that it is nonempty w (w^*)-compact and convex being understood). We note in passing that for those of an intuitionist pe more constructive approaches to the existence of such sets are possible [see G-K, for example].

In particular we readily see that such a minimal invariant set C must contain more than one poi observation best expressed by $\operatorname{diam}(C) := \sup\{\|x - y\| : x, y \in C\} > 0$, and that

$C = \overline{\operatorname{co}}\, T(C)$, in the w-case, or

$C = \overline{\operatorname{co}}^{w^*} T(C)$, in the w^*-case.

Our *meta-strategy* is to find further properties of such a minimal invariant set which are known to ruled out by (geometric) properties of the space (or of the domain of T).

Lemma 2.1 Let T be a fixed point free nonexpansive map and C a minimal invariant set for T. *If* $\psi : C \to \mathbb{R}$ *satisfies*

"(i)" ψ is a w (w^*)-lower semi-continuous convex function, and

"(ii)" $\psi(Tx) \leq \psi(x)$, for all $x \in C$

then ψ is constant on C.

Proof $D := \{x \in C : \psi(x) = \inf \psi(C)\}$ is a nonempty w (w^*)-closed convex T-invariant subset of C and so by minimality $D = C$.

>From this we have the following basic observation of Brodskii and Mil'man [B-M].

Proof It suffices to verify that $\psi(x) := \operatorname{rad}(x, C)$ satisfies the hypotheses of lemma 2.1, as then constant on C with value equal to

$$\sup_{x \in C} \psi(x) = \sup_{x \in C} \sup_{y \in C} \|x - y\| = \operatorname{diam}(C)$$

To complete the proof we first note that ψ is the supremum of convex w (w^*)-lower semi-continuous fun and so is itself convex and w (w^*)-lower semi-continuous. We next show that

$$\psi(x) = \sup_{y \in \operatorname{co} T(C)} \|x - y\|.$$

In the w-case this follows immediately since $T(C) = C$. In the w^*-case $C = \overline{\operatorname{co}}^{w^*} T(C)$, so given $\epsilon > 0$ there exists a $y_\epsilon \in C$ with $\psi(x) - \epsilon \leq \|x - y_\epsilon\|$ and a net $y_\alpha \xrightarrow{w^*} y_\epsilon$ with $y_\alpha \in \operatorname{co} T(C)$. Thus,

$$\psi(x) - \epsilon \leq \|x - y_\epsilon\| \leq \liminf_\alpha \|x - y_\alpha\|$$

and so there exists a $y \in \operatorname{co} T(D)$ with $\psi(x) - 2\epsilon \leq \|x - y\|$.

It now follows by standard convexity arguments that

$$\psi(x) = \sup_{y \in T(C)} \|x - y\|,$$

and so $\psi(Tx) \leq \psi(x)$, completing the proof.

This has led to the concept of normal structure. We say that X (X^*) has *w (w^*)-normal structure* if it does not contain any nontrivial w (w^*)-compact convex diametral subsets.

For example If (e_n) denotes the usual basis in c_0, then $C := \overline{\text{co}}\{e_n\}$ is a w-compact $(e_n \xrightarrow{w} 0)$ convex diametral subset. *So c_0 fails to have w-normal structure.*

Clearly

$$w\ (w^*)\text{-normal structure} \Rightarrow w\ (w^*)\text{-fpp}$$

We can classify conditions on X (X^*) as 'strong' if they imply $w\ (w^*)$-normal structure, and 'weak' if they fail to imply $w\ (w^*)$-normal structure while still ensuring the $w\ (w^*)$-fpp for $X\ (X^*)$. In section we will survey some of the known strong conditions. Weak conditions usually involve 'asymptotic structure in $X\ (X^*)$ and are best dealt with after we have considered Banach space ultraproducts in section First, however, in preparation for later we consider the remaining important structure known for invariant set CT for

Alongside the diametrality of C, another important observation has been the existence in C *approximate fixed point sequence* for T. That is, a sequence $(a_n) \subset C$ with $\|a_n - Ta_n\| \to 0$ (to obtain such a sequence, choose $x_0 \in C$ and take a_n to be the unique fixed point of the strict contraction $(1-\frac{1}{n})T$ guaranteed courtesy of the Banach contraction mapping principle).

Now, let (a_n) be a given approximate fixed point sequence for T in the minimal invariant $w\ (w^*)$-compact convex set C. For each subsequence (x_k) of (a_n) consider

$$\psi(x) := \limsup_k \|x - x_k\|.$$

If ψ is $w\ (w^*)$-lower semi-continuous, then lemma 2.1 applies to give ψ has constant value c on C. If $x_{k_\alpha} \xrightarrow{w\,(w^*)} x_0$ we have

$$\begin{aligned} \text{diam}\,(C) \geq c = \psi(x) &\geq \limsup_\alpha \|x - x_{k_\alpha}\| \\ &\geq \liminf_\alpha \|x - x_{k_\alpha}\| \\ &\geq \|x - x_0\|, \quad \text{for all } x \in C. \end{aligned}$$

So, since C is diametral, we have $c = \text{diam}\,(C)$, and we have arrived at the fundamental lemma of Goebel [G] and Karlovitz [Ka]:

$$\lim_n \|x - a_n\| = \text{diam}\,(C).$$

When this holds we refer to (a_n) as a *diameter sequence* for C.

In the w-case this is always the case (this was the result of Goebel and Karlovitz), as w-lower continuity is ensured for the functions ψ defined above.

In the w^*-case this may not be the case.

Example In $\ell_\infty = \ell_1^*$ for

$$a_n := (\underbrace{0, 0, \cdots, 0}_{n\ 0's}, -1, -1, \cdots)$$

and $\psi(x) := \limsup_n \|x - a_n\|$ we have for

$$x_n := (1, \cdots, 1, 0, 0, \cdots) \xrightarrow{w^*} x_\infty := (1, 1, 1, 1, \cdots)$$

that $\psi(x_n) = 1$ while $\psi(x_\infty) = 2$. *So ψ is not w^*-lower semi-continuous. This example along with an example in a separable dual are given in [S92b].*

The question of when such ψ are w^*-lower semi-continuous, and so a w^*-Goebel/Karlovitz lemma pertains, has been taken up by M. A. Khamsi [Kh] and [S92b], who established the result for stable

(and duals with shrinking strongly monotone basis) and w^*-sequentially orthogonal dual lattices: that is, Banach lattices for which

$$f_n \xrightarrow{w^*} 0 \implies \| |f_n| \wedge |f| \| \longrightarrow 0, \quad \text{for all } f \in X^*.$$

The extension of the Goebel/Karlovitz lemma to a wider class of dual spaces is an open problem.

The Goebel/Karlovitz lemma appears to endow minimal invariant sets with a richer structure mere diametrality: namely, the existence of a diametrizing sequence. Unfortunately the existe nontrivial diameteral set implies the existence of a diametral set containing a diameterizing s see this, suppose D is a closed convex diametral set, $d := \operatorname{diam}(D) > 0$. Starting with any $x_1 \in D$ we may inductively choose a sequence (x_n) so that $\| x_{n+1} - \frac{1}{n} \sum_1^n x_k \| > d - 1/n^2$. Then an easy calculation shows that $\operatorname{dist}(x_{n+1}, \{x_k\}_{k=1}^n) \longrightarrow d$. Thus (x_n) is diameterizing for $C := \overline{\operatorname{co}}\{x_k\}_{k=1}^\infty$.

If D is w-compact then so too is C. In general, we can not reach such a conclusion in the w^*-c however, D is norm separable, as would be the case in a separable dual, we may use the following de C. Lennard [Le]. Let (y_n) be a dense sequence in D, and modify the construction of the sequence (x_n) choosing it so that

$$\| x_{n+1} - \frac{1}{2n} \sum_{k=1}^{n} (x_k + y_k) \| > d - \frac{1}{n^2},$$

then (x_n) is diameterizing for all points of D. In particular it is diameterizing for $C := \overline{\operatorname{co}}^{w^*}\{x_k\}_{k=1}^\infty$.

The following characterization of normal structure due to T. Landes [La84] will be of subsequ portance.

Lemma 2.3 X contains a sequence which is diameterizing for its closed convex hull if and only if X contains a sequence (x_n) for which there is $c > 0$ with $\lim_n \|x - x_n\| = c$ for all $x \in \overline{\operatorname{co}}\{x_n\}_{n=1}^\infty$.

Proof Starting with $x_1 := x_1$ extract a subsequence (x_{n_k}) so that $\|x_{n_k} - x_{n_j}\| \leq (1 + 1/k)c$, for $j \leq k$. Put

$$z_k := \frac{k}{k+1} x_{n_{k+1}} + \frac{1}{k+1} x_1$$

and let

$$C := \overline{\operatorname{co}}\{z_k\}_{k=1}^\infty.$$

A calculation shows that $\|z_n - z_k\| \leq c$ (take $n > k$), so $\operatorname{diam}(C) = c$ and (z_k) is diameterizing for C.

The converse is immediate.

Note: If in addition (x_n) is weak convergent then C is w-compact.

The discussions preceding lemma 2.3 make it clear that to establish the w (w^*)-fpp from weak conc where w (w^*)-normal structure fails, will require delicate arguments; arguments which make exp of the fact that the diameterizing sequence is an approximate fixed point sequence for a fixed po nonexpansive mapping. Such arguments are considered in section 5. In the next section we consid of the known strong conditions.

3. Strong end games

In this section we will be concerned with the following conditions on a Banach space X, whi su cient for w-normal structure.

" (UC)" *Uniform convexity*: For every $\epsilon > 0$ there is a $\delta > 0$ such that $x, y \in B_X$ and $\|(1/2)(x+y)\| > 1-\delta$ implies $\|x-y\| \leq \epsilon$. This may be rephrased as; if C is a closed convex subset of B_X with $\operatorname{diam}(C) \geq \epsilon$ then $C \cap (\delta B_X) \neq \emptyset$. It is well known (Mil'man/Pettis theorem) that (UC) implies superreflexivity. Indeed it implies superreflexivity, and every superreflexive space admits an equivalent (UC) norm.

" (ϵ_0-InQ)" *ϵ_0-inquadrate*: The same as (UC), but only for ϵ greater than or equal to some given $\epsilon_0 \in [0, 2)$. (ϵ_0-InQ) is sufficient for superreflexivity.

" (ϵ_0-UKK)" *ϵ_0-uniformly Kadec-Klee*: For every $\epsilon > 0$ there is a $\delta > 0$ such that, if $x_n \xrightarrow{w} 0$, $\limsup_n \|x_n\| \leq 1$, and $\operatorname{sep}(x_n) := \inf\{\|x_m - x_n\| : m \neq n\} \geq \epsilon$, implies $\|x_n\| \leq 1-\delta$. This may be rephrased as; if C is a w-compact convex subset of B_X with a (sequential) measure of noncompactness $\operatorname{nc}(C) := \sup\{\operatorname{sep}(x_n) : (x_n) \subset C\} \geq \epsilon$ then $C \cap (\delta B_X) \neq \emptyset$. This generalization of R. Huff's [H] notion of uniform Kadec-Klee was introduced by van Dulst and Sims [vanD-S], where it was shown to imply that the Chebyshev centre $\{x \in C : \operatorname{rad}(x, C) = \inf_{y \in C} \operatorname{rad}(y, C)\}$ is norm compact. Whether or not the asymptotic centre is norm compact in the same circumstances remains an open question.

" (GGLD)" *Generalized Gossez-Lami Dozo property*: Whenever (x_n) is a w-null sequence which is not norm convergent we have

$$\liminf_n \|x_n\| < \lim_m \sup_n \|x_m - x_n\|,$$

(GGLD) was introduced by A. Jimenez-Melado [JM].

" (ass-P)" *Asymptotic–P*: Whenever (x_n) is a w-null sequence which is not norm convergent we have

$$\liminf_n \|x_n\| < \operatorname{diam}_a\{x_n\},$$

where $\operatorname{diam}_a\{x_n\} := \lim_n \operatorname{diam}\{x_k\}_{k=n}^{\infty}$ is the *asymptotic diameter* of the sequence (x_n). Asymptotic–P was introduced in [S-S].

" (P)" *Property–P*: Whenever (x_n) is a nonconstant w-null sequence we have

$$\liminf_n \|x_n\| < \operatorname{diam}\{x_n\}$$

Property–P was first considered by K. K. Tan and H. K. Xu [T-X].

" (O)" *Opial's condition*: Whenever $x_n \xrightarrow{w} 0$ and $x \neq 0$ we have

$$\limsup_n \|x_n\| \ < \ \limsup_n \|x_n + x\|.$$

This condition was introduced by Z. Opial [O] and shown to imply w-normal structure by J. P. Gossez and E. Lami Dozo [G-L].

" (WO)" *Weak Opial*: Whenever (x_n) is a w-null sequence which is not norm convergent we have

$$\liminf_n \|x_n\| \ < \ \sup_m \limsup_n \|x_m - x_n\|,$$

this is equivalent to requiring that there exist $x \neq 0$ with $\limsup_n \|x_n\| < \limsup_n \|x - x_n\|$. This weakening of the Opial condition was introduced by Tingley [T].

Most of the following implications are clear from the respective definitions.

$(UC) \Rightarrow (\epsilon_0\text{-InQ})\ 0 \le \epsilon_0 < 1 \Rightarrow (\epsilon_0\text{-UKK}),\ 0 \le \epsilon_0 < 1 \Downarrow$ (GGLD) $\Downarrow$ (ass-P) $\Downarrow$ (O) $\Rightarrow$ (WO) $\Rightarrow$ (P) $\Downarrow$ n. str.

It is also clear that (GGLD) $\Rightarrow$ (WO).

To see that, for $0 \le \epsilon_0 < 1$, (ϵ_0-UKK) implies (ass-P), suppose there is $x_n \xrightarrow{w} 0$ with $\liminf_m \|x_m\| = \operatorname{diam}_a \{x_n\} = 1$. Then since we can choose m so that $\|x_m\|$ is arbitrarily close to 1 and $\limsup_n \|x_m - x_n\| = \|x_m\|$, we can extract a subsequence (x_{n_k}) with sep $(x_{n_k}) > \epsilon_0$ and choose an m_0 so that $\|x_{n_k} - x_{m_0}\| \ge \|x_{m_0}\| > 1 - \delta$. But then $y_k := x_{m_0} - x_{n_k}$ defines a sequence which contradicts (ϵ_0-UKK). Whether or not (ϵ_0-UKK), for $\epsilon_0 \ge 1$, implies w-normal structure remains unanswered. The result does, however, follow if an additional property (WORTH) [S94].

We will show that the properties (GGLD) and (ass-P) coincide, as do (WO) and (P). Apart from the equivalences all the other implications given above are strict. In most cases this follows from examples. For instance, all the spaces L_p, $p \ne 2$ fail to have (O), but for $p \ge 3$ satisfy $\limsup_n \|x_n\|_p^p \le (1/2) \limsup_m \limsup_n \|x_m - x_n\|_p^p$ whenever $x_n \xrightarrow{w} 0$, and so these spaces are 'WOfull'.

The space c_0 equivalently renormed by

$$\|(\xi_n)\| := \|(\xi_n)_\infty\| + \sum_n \frac{|\xi_n|}{2^n},$$

was considered by A. Jimenez-Melado [JM]. It enjoys (O) [van D], but lacks (ass-P). Thus, while the condition implies property (P) it fails to imply any of the stronger conditions listed.

To separate w-normal structure and (P) is more delicate. We first note the following two propositions.

Proposition 3.1 A Banach space has (WO) if and only if it has (P).

Proof We need only prove (P) $\Rightarrow$ (WO). Suppose X fails to have (WO), then we can find $x_n \xrightarrow{w} 0$ with $\|x_n\| \to 1$, but $\limsup_n \|x_m - x_n\| \le 1$ for all m. Construct from (x_n) the z_k's as in the proof of Landes' result lemma 2.3. Then $z_k \xrightarrow{w} 0$, $\|z_k\| \to 1$, and $\operatorname{diam}\{z_k\} \le 1$, so X fails to have (P).

Proposition 3.2 If X has w-normal structure and satisfies the nonstrict Opial condition:

$$\limsup_n \|x_n\| \le \limsup_n \|x + x_n\|, \quad \text{for all } x \in X,$$

whenever $x_n \xrightarrow{w} 0$, then X has (P).

Proof Suppose X fails (P), and hence (WO), then there exists a sequence $x_n \xrightarrow{w} 0$, with $\|x_n\| \to 1$, $\limsup_n \|x - x_n\| \le 1$, for all $x \in C := \overline{\operatorname{co}}\{x_n\}$. But, by the nonstrict Opial condition $\limsup_n \|x - x_n\| \ge \lim_n \|x_n\| = 1$. It follows that $\operatorname{diam}(C) = 1$ and (x_n) is a diameterizing sequence for C. Thus X fails w-normal structure.

To separate (P) from w-normal structure we make use of results by T. Landes. Landes [La86] introduces the *weak sum property* as follows.

Say a sequence (x_n) is *limit affine* if $\psi(x) := \lim_n \|x - x_n\|$ exists and is a constant on $\overline{\operatorname{co}}\{x_n\}$ (we can in fact drop the existence assumption and replace lim by lim sup, and require only that $\psi(x_n)$ is nondecreasing). A space X has the *weak sum property* (WSP) if the only w-convergent nondecreasing limit affine sequences

are the constant sequences. The name derives from Landes' result that the property is preserve the taking of nite sums, and is the weakest Banach space property so preserved which implies structure.

It is readily seen that (P) implies (WSP): If $x_n \xrightarrow{w} 0$ is a nonconstant nondecreasing and limit a ne sequence, then the function ψ it de nes is a ne norm continuous, and hence also w-continuous, on $\overline{\{x_n\}}$. Further $l := \lim \psi(x_n)$ exists. It follows, since ψ is nondecreasing, that $\|x_n\| = \psi(0) = l \geq \lim_n \|x_m - x_n\|$, contradicting (WO) and hence (P).

Landes considers two renormings of c_0

$$X_1 := (c_0, \|\cdot\|') \quad \text{and } X_2 := (c_0, \|\cdot\|'')$$

where

$$\|(x_n)\|' := \sup_{i<j} \left| \sum_{\substack{k=1 \\ k \neq i}}^{j-1} \frac{2}{3^k} x_k + (1 - 1/3^i) x_j \right|$$

and

$$\|(x_n)\|'' := \|(x_n)\|' \sup_j \left| \sum_k f_k x_k \right|,$$

where

$$f_k = \{ 1/3^k, for$$

k<j, $(1 - 1/3^j)$ for k=j, 0 for k>j

X_1 has (WSP), but not (P): $e_n \xrightarrow{w} 0$, $\|e_n\|' = 1$, and diam $(e_n) = 1$, while, X_2 has w-normal structure, but not (WSP).

Thus, in general, w-normal structure $\not\Rightarrow$ (WSP) $\not\Rightarrow$ (P).

Our nal result is the equivalence of (GGLD) and (ass-P).

Proposition 3.3 A Banach space has (GGLD) if and only if it has (ass-P).

Proof It is su cient to prove the implication (ass-P) $\Rightarrow$ $\liminf_n \|x_n\|$ whenever (x_n) is a w-null sequence which does not converge in norm. As then (ass-P) $\Rightarrow$ (GGLD) follows, and the rev implication is clear.

Our proof is based on a technique explored by T. Benavides. To obtain a contradiction suppose exists a sequence $x_n \xrightarrow{w} 0$, with $\|x_n\| \to 1$ and sep $(x_n) \leq 1$. Given $\epsilon > 0$ de ne

$$A := \{ \{x_n, x_m\} : \|x_m - x_n\| \geq 1 + \epsilon\}, \quad \text{and}$$
$$B := \{ \{x_n, x_m\} : \|x_m - x_n\| < 1 + \epsilon \text{ and } x_m \neq x_n\}.$$

Then $A \cup B$ is the set of all two element subsets of $\{x_n\}_{n=1}^{\infty}$ and Ramsey's theorem ensures the existence of a subsequence (x_{n_k}) with $\{ \{x_{n_k}, x_{n_j}\} : x_{n_k} \neq x_{n_j}\}$ either contained entirely in A, or entirely in B. Now sep $(x_n) \leq 1$, so the above set lies entirely in B. That is, $\|x_{n_k} - x_{n_j}\| < 1 + \epsilon$, for all $k < j$, and so diam $(x_{n_k}) \leq 1 + \epsilon$.

Repeated application of this, with $\epsilon = 1/n$, and a diagonalization argument, yields a subsequence (z_n) with diam $(z_n) \leq 1$, but $\|z_n\| \to 1$, so X fails to enjoy (ass-P).

4. Ultra-techniques

In this section we develop the Banach space ultrapower and initiate its use as a tool for stud w-fpp. Throughout I will denote an index set, for our purposes usually N. For a more extensive and deta development than is possible here the reader is referred to [G-K], [A-K], or [S82a].

A *filter* on I is a nonempty family of subsets $F \subseteq 2^I$ satisfying

" (i)" F is closed under taking supersets: if $A \in F$ and $A \subseteq B \subseteq I$ $\Rightarrow$ $B \in F$.

" (ii)" F is closed under finite intersections: if $A, B \in F$ $\Rightarrow$ $A \cap B \in F$.

We say F is *countably complete* if it is closed under countable intersections.

" (1)" 2^I

" (2)" The Frechet filter $\{A \subseteq I : I \backslash A \text{ is finite}\}$

" (3)" For $i_0 \in I$, $F_{i_0} := \{A \subseteq I : i_0 \in A\}$. Filters of the form F_{i_0} for some $i_0 \in I$ are called *trivial*, and *non-free* filters.

A filter F is *proper* if it does not equal the power set of I, 2^I. Equivalently, if $\emptyset \notin F$. Additional conditions are: has the finite intersection property.

Henceforth by filter we will mean proper filter.

An *ultrafilter* U on I is a filter on I which is maximal with respect to ordering of filters on I by inclus that is, if $U \subseteq F$ and F is a filter on I, then $F = U$. Zorn's lemma ensures that every filter has an ext to an ultra filter.

Lemma 4.1 A [proper] filter $U \subset 2^I$ is an ultra filter on I if and only if for every $A \subseteq I$ precisely one of A or $I \backslash A$ is in U.

As a consequence of this lemma: For an ultra filter U on I, if $I = A_1 \cup A_2 \cup \cdots \cup A_n$ then at least one of the sets $A_1, A_2, \cdots, A_n$ is in U, and an ultra filter is nontrivial (free) if and only if it contain subsets.

It will henceforth be a standing assumption that all the filters and ultra filters with which nontrivial.

An ultra filter U is countably incomplete if and only if there exist elements $A_0, A_1, \cdots, A_n, \cdots$ with

$$I = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots \quad \text{and} \quad \cap_{n=0}^{\infty} A_n = \emptyset.$$

Countably incomplete ultra filters are particularly convenient, as we shall see inductive and dia arguments are readily extended to them. Every ultra filter U over N is necessarily countably inc $(A_n := \{n, n+1, n+2, \cdots\} \in U)$.

For a Hausdorff topological space T, ultra filter U on I, and $\{x_i\}_{i \in I} \subset T$ we say

$$U - \lim x_i \equiv T - U - \lim x_i = x_0$$

if for every neighbourhood N of x_0 we have $\{i \in I : x_i \in N\} \in U$. Limits along U are unique and satisfy all the usual limit theorems when T is a linear topological space. If T is compact then $U - \lim x_i$ e

all $x = (x_i)_{i \in I}$. If U is on $\mathbb{N}$ and (x_n) is a bounded sequence in $\mathbb{R}$ then

$$\liminf_n x_n \leq U - \lim x_n \leq \limsup_n x_n$$

For a Banach space X and ultrafilter U on I we can form the substitution space

$$\ell_\infty(X) := \{(x_i)_{i \in I} : \|(x_i)\| = \sup_{i \in I} \|x_i\| < \infty\}.$$

$$N_U(X) := \{(x_i)_{i \in I} : U - \lim \|x_i\| = 0\}$$

is a closed linear subspace of $\ell_\infty(X)$.

The *Banach space ultrapower* of X over U is defined to be the quotient space

$$(X)_U := \frac{\ell_\infty(X)}{N_U(X)}$$

with elements denoted $[x_i]_U$ and the quotient norm cannonically given by

$$\|[x_i]_U\| = U - \lim \|x_i\|.$$

The mapping

$$J : X \longrightarrow (X)_U : x \longmapsto [x] := [x_i]_U \text{ where } x_i = x, \text{ for all } i \in I$$

is an isometric embedding of X into $(X)_U$.

For a closed bounded convex subset C of X and nonexpansive mapping $T : C \longrightarrow C$ we define in $(X)_U$

$$\tilde{C} := \{[c_i]_U : c_i \in C, \text{ for all } i \in I\}.$$

$\tilde{C}$ is convex subset, with $\operatorname{diam}(\tilde{C}) = \operatorname{diam}(C)$, containing the isometric copy $J(C)$ of C, and on which

$$\tilde{T} : \tilde{C} \longrightarrow \tilde{C} : [c_i]_U \longmapsto [Tc_i]_U,$$

where the representatives c_i are chosen from C, is a well defined nonexpansive mapping which leaves
invariant.

Let U be an ultrafilter over $\mathbb{N}$. Then for $\tilde{C}$ and $\tilde{T}$ constructed as above we have the following resu

Proposition 4.2 If (a_n) is an approximate fixed point sequence for T, then $[a_n]_U$ is a fixed point of $\tilde{T}$. *So $\tilde{T}$ always has fixed points in $\tilde{C}$.*

Conversely, from a fixed point (indeed an approximate fixed point sequence) for $\tilde{T}$ in $\tilde{C}$ we can re extract an approximate fixed point sequence for T. Although we will not make use of it, it is worth the following significant observation of B. Maurey [M] (see [El], or [A-K], for more details).

Lemma 4.3 'Between' any two fixed points a and b of $\tilde{T}$ in $\tilde{C}$ there is a fixed point c with

$$\|a - c\| = \|c - b\| = \frac{1}{2}\|a - b\|.$$

That is, the fixed point set of $\tilde{T}$ is metrically convex.

The following generalization of the Goebel/Karlovitz lemma, due to P. K. Lin [Lin] has proved b establishing the fpp in the presence of weak conditions. The proof of this and the next result illu 'diagonal' arguments over an incomplete ultrafilter.

Lemma 4.4 Suppose C is a w-compact minimal invariant set for T. If (x_n) is an approximate fixed point sequence for T in C then

$$\lim_n \|x_n - Jx\| = diam\,(C), \quad \text{for all } x \in C.$$

Proof Suppose this were not the case. Without loss of generality we may take $diam\,(C) = diam\,(C)$ and by passing to a subsequence if necessary assume that $\|x_n - Tx_n\| < 1/n$.

Then there are $\epsilon_0 > 0$, $x_0 \in C$, and $n_0 \in \mathbb{N}$ with

$$\|x_n - Jx_0\| < 1 - \epsilon_0, \quad \text{for all } n > n_0.$$

Let $x_n = [a_m^n]_U$, with $a_m^n \in C$, and define

$$A_n := \{m : \|a_m^n - x_0\| < 1 - \epsilon_0/2\},$$

and

$$B_n := \{m : \|a_m^n - Ta_m^n\| < 2/n\}.$$

Then A_n and B_n are in U.

Put $m_0 = 0$ and for $n \in \mathbb{N}$ inductively choose $m_n \in B_n \cap A_n \cap \{m_{n-1} + 1, m_{n-1} + 2, \cdots\} \in U$. Then the sequence $(a_{m_n}^n)$ is such that

$$\|a_{m_n}^n - Ta_{m_n}^n\| < 2/n.$$

That is, $(a_{m_n}^n)$ is an approximate fixed point sequence for T in C. But,

$$\|a_{m_n}^n - x\| < 1 - \epsilon_0/2,$$

an observation which is difficult to reconcile with the fact that $(a_{m_n}^n)$ is, by the Goebel/Karlovitz diameterizing for C.

Proposition 4.5 The set C in $(X)_U$ is closed. *Hence when X is a superreflexive space C is w-compact.*

Proof While the proof is true for any ultrafilter U, we will only prove it in the case when U is countably incomplete with $I = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$, each $A_n \in U$, and $\cap A_n = \emptyset$.

Suppose $[t_i^1]_U, [t_i^2]_U, \cdots$ is a sequence of points in C, with $t_i^j \in C$ for each i, which converges to $[x_i]_U \in (X)_U$. By passing to a subsequence if necessary we may without loss of generality assume that

$$\|[t_i^m] - [x_i]\| = U - \lim_i \|t_i^m - x_i\| < \frac{1}{m}.$$

For each $m \in \mathbb{N}$ let

$$B_m := \{i \in I : \|t_i^m - x_i\| < 2/m\} \cap A_m \in U.$$

and put $B_0 := I$ and $t_i^0 := 0$, then

$$I = B_0 \supset B_1 \supset B_2 \supset \cdots \supset B_m \supset \cdots, \quad \text{and } \cap_{m=0}^{\infty} B_m = \emptyset.$$

>From this it follows that for each $i \in I$ there is a unique m such that $i \in B_m \setminus B_{m+1}$. Define $y_i := t_i^m$, for this m, in particular then $y_i \in C$.

Now, given any $m \in \mathbb{N}$, for each $i \in B_m$ there is a unique $p \geq m$ with $i \in B_p \setminus B_{p+1}$. thus,

$$\|y_i - x_i\| = \|t_i^p - x_i\| < 2/p \leq 2/m,$$

and so

$$\{i \in I : \|y_i - x_i\| < 2/m\} \supseteq B_m \in U.$$

We therefore have that $U - \lim \|y_i - x_i\| = 0$, which yields the desired conclusion that $[x_i] \in \tilde{C}$.

5. Weak ends

The results developed in the last section have been used to establish the w-fpp for a variety of conditions on the space X. For example, B. Maurey [M] established the w-fpp for c_0, and for deeper arguments for the re exive subspaces $L_p[0,1]$ and the Hardy space H_1 (see also, [E] and [A-K] for more details). J. Borwein and Sims [B-S] generalized Maurey's c_0 result to obtain the w-fpp for Banach with a Riesz angle

$$\alpha(X) := \sup\{\| |x| \vee |y| \| : x, y \in B_X\} < 2$$

and for which

$$\liminf_m \liminf_n \| |x_n| \wedge |x_m| \| = 0, \quad \text{whenever } x_n \xrightarrow{w} 0.$$

Lattices with this last property were referred to as *weakly orthogonal* Banach lattices. A formally stronger version of weak orthogonality:

$$\liminf_n \| |x_n| \wedge |x| \| = 0, \quad \text{whenever } x_n \xrightarrow{w} 0 \text{ and } x \in X,$$

was shown to imply the w-fpp by Sims [S88], [S92a]. This was obtained using arguments similar to those developed by P. K. Lin [Lin] to prove the w-fpp for Banach spaces with a 1-unconditional basis.

Several more 'geometric' variants of these conditions have been introduced. For instance A. Melado and E. Llorens-Fuster [JM-LF] considered the property of orthogonal convexity, gave examples of orthogonally convex spaces, and showed that it entails the w-fpp . A Banach space X is said to be *orthogonally convex* if for every w-null sequence (x_n) with

$$D(x_n) := \limsup_m \limsup_n \|x_m - x_n\| > 0$$

there exists $\beta > 1/2$ such that

$$\limsup_m \limsup_n \sup\{\|z\| : \|z - x_m\|, \|z - x_n\| \leq \beta \|x_m - x_n\|\} < D(x_n).$$

The signi cance of a Riesz angle was transported to general Banach spaces by J. Garcia-Falset an ultra lter over $\mathbb{N}$ de ne

$$R(X) := \sup\{U - \lim \|x + x_n\| : x, x_n \in B_X \text{ and } x_n \xrightarrow{w} 0\}.$$

For example, $R(c_0) = 1$, while $R(L_p) = 2$, in general $1 \leq R(X) \leq 2$.

Garcia-Falset [GF94] showed that a Banach space X has the w-fpp if $R(X) < 2$ and it satis es nonstrict Opial condition:

$$U - \lim \|x_n\| \leq U - \lim \|x_n + x\|, \quad \text{whenever } x \neq 0 \text{ and } x_n \xrightarrow{w} 0.$$

Recently he has proved the result without requiring the nonstrict Opial condition [GF95].

We illustrate how the machinery of section 4. may be used by proving the special case when $R(X)$

Proposition 5.1 If X is a Banach space with $R(X) = 1$ then X has the w-fpp.

Proof We first observe that $R(X) = 1$ implies the nonstrict Opial condition. If this were not the could find a sequence $x_n \xrightarrow{w} 0$, with $U-\lim\|x_n\| = 1$ and $U-\lim\|x+x_n\| < 1$ for some $x \neq 0$. Since $x+x_n \xrightarrow{w} x$ we have $\|x\| \leq U-\lim\|x+x_n\| < 1$. From the definition of $R(X)$ we therefore have $U-\lim\|x-x_n\| \leq 1$. But, then $U-\lim\|x_n\| \leq U-\lim(\|x+x_n\| + \|x-x_n\|/2)$, a contradiction.

Now, suppose X fails the w-fpp. Let C be a w-compact convex set with $\operatorname{diam}(C) = 1$ which is a minimal invariant set for the nonexpansive mapping T, and let (a_n) be a w-null approximate fixed point sequence for T. For an ultrafilter U over $\mathbb{N}$ let $\tilde{C}$ and $\tilde{T}$ be defined as in section 4. Define

$$W := \{\tilde{w} \in \tilde{C} : \|\tilde{w} - [a_n]\| \leq 1/2 \text{ and } \exists\, x \in C \text{ with } \|\tilde{w} - Jx\| \leq 1/2\}.$$

Then, W is a $\tilde{T}$-invariant, closed convex nonempty (as $(1/2)[a_n] \in W$) set, which, by the standard co tion using Banach contraction mapping principle, contains an approximate fixed point sequence fo by the generalized Goebel/Karlovitz lemma (lemma 4.4) $\sup\{\|\tilde{w}\| : \tilde{w} \in W\} = 1$.

On the other hand, let $\tilde{w} = [w_n] \in W$, where without loss of generality $w_n \in \mathbb{N}$, and let $v = w-U-\lim w_n$. Then, since $a_n \xrightarrow{w} 0$, the w-lower semi-continuity of the norm gives

$$(1/2) \geq \|\tilde{w} - [a_n]\| := U-\lim\|w_n - a_n\| \geq \|v\|,$$

and the nonstrict Opial condition established above gives

$$\|\tilde{w} - Jv\| := U-\lim\|w_n - v\| \leq U-\lim\|w_n - x_0\| =: \|\tilde{w} - Jx_0\| \leq 1/2,$$

for some $x_0 \in C$, which exists by the definition of W. Thus

$$\|\tilde{w}\| \leq \|\tilde{w} - Jv + Jv\| \leq R(X)\|\tilde{w} - Jv\| \vee \|v\| \leq 1/2.$$

A contradiction establishing the result.

The above proof is typical of those for many of the results mentioned above. A numeric contrac is arrived at, here $1 \neq 1/2$, and by carefully analysing the proof the gap, here between 1/2 and be exploited to establish the w-fpp for spaces whose Banach-Mazur distance from a space satisf conditions is not too great. We will not pursue such stability issues. The interested reader is di papers cited above.

As mentioned above, proposition 5.1 has been generalized to only requiring $R(X) < 2$. Sinc Banach lattice $R(X)$ (≤ 2), this substantially extends the result of Borwein and Sims, by showing tha the presence of a Riesz angle condition the assumption of weak-orthogonality is unnecessary. A gene in another direction is encompassed by the following.

For the rest of this section U will be an ultrafilter on $\mathbb{N}$.

A *weakly null type* on a Banach space X is a function of the form

$$\psi_{(x_n)}(x) = U-\lim_n\|x - x_n\|,$$

where (x_n) is a w-null sequence. We say $\psi_{(x_n)}$ is nontrivial if $\psi_{(x_n)}(0) \neq 0$ — that is, if $\|x_n\| \to 0$. If X is a separable space we may replace (x_n) by a subsequence so that $\psi_{(x_n)}(x) = \lim_n\|x - x_n\|$, for all $x \in X$.

As we have seen, weakly null types play an important rôle in metric fixed point theory. Note example, our proof of the Brodskii-Mil'man theorem and the Goebel/Karlovitz lemma in section 2, ma

the conditions considered in section 3 (for instance, Opial's condition is essentially a stateme
the behaviour of weakly null types), the de nition of $R(X)$, and the proof for proposition 5.1 given above.

Recently N. Kalton [K] introduced *property (M)*: Weakly null types are constant on spheres about the origin. That is, for $x_n \xrightarrow{w} 0$ the weakly null type $\psi_{(x_n)}(x) = U - \lim_n \|x - x_n\|$ is a function of $\|x\|$ only.

Examples of spaces with property (M) include $L_p[0,1]$ for $1 \le p < \infty$, indeed all the Orlicz sequence spaces with the Δ_2-condition. Subspaces of $L_1[0,1]$ with property (M) have the uniform Opial property of Prus [P], and so have normal structure (see Kalton and Werner [K-W], for details).

Property (M) was one of the two essential ingredients in Kalton's characterization of those Banach spaces X for which the compact operators $K(X)$ form an M-ideal in the algebra of all bounded linear operators, $L(X)$. That is,

$$L(X)^* = \left(K(X)^\perp \oplus V\right)_1, \quad \text{for some closed subspace } V.$$

Asvald Lima [Li] e ectively proved that the dual of such a space is weak*-uniformly Kadec-Klee (U
and hence has weak* normal structure (see, [S82b], for more details). This, combined with the r^ole
null types noted above, makes it natural to inquire into connections between property (M), weak
structure, and the w-fpp .

Lemma 5.2 Each of the following properties implies the one below it.

" (i)" X has property (M).

" (ii)" If $x_n \xrightarrow{w} 0$ and $\|x\| \le \|y\|$ then $\psi_{(x_n)}(x) \le \psi_{(x_n)}(y)$.

" (iii)" X satis es the non-strict Opial condition.

Proof Only (i) $\Rightarrow$ (ii) requires proof, and from the de nition of property (M) it is enough to s
that $\psi_{(x_n)}(tx)$ is an increasing function of t on $[0,\infty)$. To see this, note that for $0 < t_1 < t_2$ th
exists $\beta \in (0,1)$ such that $t_1 x = \beta(-t_2)x + (1-\beta)t_2 x$ and so, since $\psi_{(x_n)}$ is convex and by property (M)
$\psi_{(x_n)}(-t_2 x) = \psi_{(x_n)}(t_2 x)$, we have $\psi_{(x_n)}(t_1 x) \le \beta\psi_{(x_n)}(-t_2 x) + (1-\beta)\psi_{(x_n)}(t_2 x) = \psi_{(x_n)}(t_2 x)$.

Proposition 5.3 Let X be a Banach space with property (M). *Then X has w-normal structure if and only if there is no nontrivial weakly null type which is identically equal to 1 on B_X.*

Proof ($\Leftarrow$) Suppose X fails to have w-normal structure then B_X contains $x_n \xrightarrow{w} 0$ with $\lim_n \|x - x_n\| =$
1, for all $x \in \{x_n\}_{n=1}^\infty$. In particular, since $0 \in \overline{\{x_k\}_{k=1}^\infty}$, we have $\|x_n\| \to 1$.

Thus, $\psi_{(x_n)}(0) = 1$ and $\psi_{(x_n)}(x) = 1$, for all Further, since $\|x_m\| \to 1$, it follows from lemma 5.2
that $\psi_{(x_n)}$ equals 1 on the open unit ball, and hence by continuity on B_X.

($\Rightarrow$) This follows immediately from lemma 2.3.

As a consequence of this lemma we have the following.

Theorem 5.4 A Banach space X with property (M) has weak normal structure if there exists a point $x_0 \in S_X$ at which the relative weak and norm topologies agree.

Proof Suppose X fails to have weak normal structure. Let $\psi_{(x_n)}$ be the weakly null type of propos
5.3, and let $y_n := x_0 - x_n$. Then, $y_n \xrightarrow{w} x_0$ and $\liminf_n \|y_n\| \le \psi_{(x_n)}(x_0) = 1$, so there is a subsequence

with $\|y_{n_k}\| \to 1$. But, then (y_{n_k}), and hence (x_{n_k}), is norm convergent, a difficult thing for a diametrizing sequence to achieve.

Corollary 5.5 If X has property (M) and satisfies any of the following then X has weak normal structure.

- "(i)" X has the Kadec-Klee property (the relative weak and norm topologies agree on S_X).
- "(ii)" X is reflexive.
- "(iii)" X has the Radon-Nikodym property.
- "(iv)" X has the point of continuity property: for every weakly closed bounded subset A, the map (A, weak) to (A, norm) has at least one point of continuity, see [E-W] for details.

A dual property to (M), property (M^*) is defined in X^* by requiring that

$$\psi_{(f_n)} : X^* \longrightarrow \mathbb{R}^+ : f \longmapsto U - \lim_n \|f - f_n\|$$

be a function of $\|f\|$ only, whenever $f_n \overset{w^*}{\to} 0$. Since (M^*) implies X^* has the Radon-Nikodym property we conclude that if X^* has property (M^*) then X^* has w^*-normal structure and hence the w^*-fpp.

It follows from the work in Kalton [K] Lemma 3.6 and the discussion preceding it that if X has property (M) and $c_0 \not\hookrightarrow X$ then X has w-normal structure. Thus, for spaces with property (M) the presence of c_0 is the only impediment to weak normal structure. It is not however an impediment for the w-fpp.

Directly after these lectures were given in Seville, J. Garcia-Falset and myself succeeded in following, using an adaptation of the methods illustrated in proposition 5.1.

Theorem 5.6 Let X be a Banach space with property (M) then X has the w-fpp (see, [GF-S], for details).

It can also be seen that if X is a separable, σ-order complete Banach lattice which admits an equivalent norm with property (M), then X has the w-fpp.

We close this section with an observation by T. Dalby which justifies our earlier claim that this represents a generalization of proposition 5.1.

proposition 5.7 For a Banach space X, if $R(X) = 1$ then X has property (M).

Proof We have seen (proof of proposition 5.1) that $R(X) = 1$ implies the nonstrict Opial condition. Now, let $x_n \overset{w}{\to} 0$ and suppose x, y are such that $\|x\| = \|y\| \neq 0$, then

$$\|y\| = \|x\| \leq \liminf_n \|x + x_n\| \leq \psi_{(x_n)}(x).$$

Also, by the nonstrict Opial condition

$$U - \lim_n \|x_n\| \leq U - \lim_n \|x + x_n\| = \psi_{(x_n)}(x).$$

Let $d := \psi_{(x_n)}(x)$, then from the definition of R(X) we have

$$U - \lim_n \|(1/d)\, y + (1/d)\, x_n\| \leq R(X) = 1.$$

That is $\psi_{(x_n)}(y) \leq \psi_{(x_n)}(x)$, and the result follows from symmetry in x and y.

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