KKM and Ky Fan Theorems in Hyperconvex Metric Spaces

Dedicated to Ky Fan

Abstract

In hyperconvex metric spaces, we introduce Knaster-Kuratowski-Mazurkiewicz mappings (in short KKM-maps). Then we prove an analogue to Ky Fan fixed point theorem in hyperconvex metric spaces.

[1]

1 Introduction

The notion of hyperconvexity is due to Aronszajn and Panitchpakdi [1] who proved that a hyperconvex space is an absolute retract, i.e. it is a nonexpansive retract of any metric space in which it is isometrically embedded. The corresponding linear theory is well developed and associated with the names of Gleason, Goodner, Kelley and Nachbin. For this linear theory the reader can refer to Lacey [8]. The nonlinear theory is still developing. Isbell [4] constructed for any metric space a natural hyperconvex hull. The recent interest into these spaces goes back to the results of Sine [11] and Soardi [12] who proved independently that fixed point property for nonexpansive mappings holds in bounded hyperconvex spaces. Since then many interesting results [2,6,7,9,10] have been shown to hold in hyperconvex spaces.
Recall also that Jawhari, Misane and Pouzet [5] were able to show that Sine and Soardi's fixed point theorem is equivalent to the classical Tarski's fixed point theorem in complete ordered sets. This happens via the notion of generalized metric spaces. Therefore, the notion of hyperconvexity should be understood and appreciated in a more abstract formulation.
In this work, we will recall and investigate some basic properties of hyperconvexity. We will also discuss Knaster-Kuratowski-Mazurkiewicz mappings (in short KKM-maps) in this setting and prove an analogue to Ky Fan's fixed point theorem which can be seen as an extension to Brouwer and Schauder's fixed point theorems. It is to our knowledge the first time that such theorems were attempted to be proved in metric space setting.

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2 Basic definitions and properties

The term hyperconvex does have some unfortunate aspects. For example, a hyperconvex subset of $\mathbf{R}^2$ need not be convex. Also convex sets can fail to be hyperconvex. Spaces as nice as the Hilbert space fails to be hyperconvex. Still one has some properties similar to convexity and others similar to compactness. More convincing analogies hold for hyperconvex sets which are ball intersections. We will start our development with the definition of hyperconvexity.

Definition 1. A metric space (M,d) is called *hyperconvex* if for any collection of points $\{x_\alpha\}$ of M and $\{r_\alpha\}$ a collection of non-negative reals such that

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$$

then

$$\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset \ .$$

Here we use $B(x,r)$ for the closed ball about $x \in M$ and of radius $r \geq 0$. This definition can be seen as a binary intersection property plus a metric convexity, that is for any $x, y \in M$ and $\alpha \in [0,1]$ there exists $z \in M$ such that $d(x,z) = \alpha d(x,y)$ and $d(y,z) = (1-\alpha)d(x,y)$. Since in linear spaces, the metric convexity holds, then this definition reduces to the binary-intersection property on balls, that is any collection of closed balls which intersects pairwise, does have a nonempty intersection. Nachbin-Kelley [8] proved that a normed linear space X satisfies this property if and only if there exists a Stonian compact set K such that $X = \mathcal{C}(K)$. For example, the spaces l^∞ and L^∞ are hyperconvex.

Definition 2. Let (M,d) be a metric space and $A \subset M$ a nonempty bounded subset. Set:

$r_x(A) = \sup\{d(x,y); y \in A\}$ for $x \in M$,

$r(A) = \inf\{r_x(A); x \in M\}$,

$R(A) = \inf\{r_x(A); x \in A\}$,

$\delta(A) = \sup\{d(x,y); x, y \in A\} = \sup\{r_x(A); x \in A\}$,

$\mathcal{C}(A) = \{x \in A; r_x(A) = R(A)\}$,

$co(A) = \bigcap\{B; B \text{ is a closed ball such that } A \subset B\}$.

$\mathcal{A}(M) = \{A \subset M; A = co(A)\}$, i.e $A \in \mathcal{A}(M)$ iff A is an intersection of balls. In this case we will say A is an admissible subset of M.

A mapping $T : M \to M$ is called nonexpansive if $d(Tx, Ty) \leq d(x,y)$ for all $x, y \in M$. A point $x \in M$ is called a fixed point of T if $T(x) = x$. The fixed point set of T will be denoted $Fix(T)$.

Proposition 1. *Let (M,d) be a metric space.*

1. *There exists an index set I and a natural isometric embedding from M into $l^\infty(I)$.*

2. *If M is hyperconvex then it is complete.*

3. *Assume that M is hyperconvex and let $A \subset M$ be a nonempty bounded subset. Then*

   3.1 $co(A) = \bigcap\{B(x, r_x(A)); x \in M\}$,

   3.2 $r_x(co(A)) = r_x(A)$ *for any* $x \in M$,

   3.3 $r(A) = \frac{1}{2}\delta(A)$,

   3.4 $r(co(A)) = r(A)$,

   3.5 $\delta(co(A)) = \delta(A)$,

   3.6 If $A \in \mathcal{A}(M)$, *then we have* $r(A) = R(A) = \frac{1}{2}\delta(A)$.

4. *M is hyperconvex iff for any metric space N which contains isometrically M, there exists a nonexpansive retract $r : N \to M$, i.e. r is nonexpansive and $r(x) = x$ for any $x \in M$.*

5. *M is hyperconvex iff for any metric space N and D which contains N metrically and any nonexpansive map $T : N \to M$, there exists an extension $T^\star : D \to M$ which is nonexpansive, i.e. $T(x) = T^\star(x)$ for any $x \in N$.*

Let us show how the natural isometric embedding of M into $l^\infty(I)$ holds. First set $I = M$ (we may take I to be any dense subset of M). Define $i : M \to l^\infty(I)$ by $i(x) = \Big(d(x,y) - d(x_0,y)\Big)_{y\in M}$, where $x_0 \in M$. It is easy to see that

$$||i(x) - i(y)|| = \sup_{z\in M} \Big|d(x,z) - d(y,z)\Big| = d(x,y)\ , \quad \text{for every } x, y \in M$$

which implies the conclusion of statement 1. Throughout this work, we will identify M with $i(M)$. If we assume that M is hyperconvex (and since hyperconvexity is preserved by isometry) then, by statement 4, there exists a nonexpansive retract $r : l^\infty(I) \to M$. Let $x_1, .., x_n \in M$ and $\alpha_1, .., \alpha_n$ non-negative numbers such that $\sum \alpha_i = 1$, then

$$r\Big(\sum_{1\le i\le n} \alpha_i x_i\Big) = \bigoplus_{1\le i\le n} \alpha_i x_i \in M$$

behaves like a convex combination of the (x_i). For example, we have

$$d\Big(\bigoplus_{1\le i\le n} \alpha_i x_i, \bigoplus_{1\le i\le n} \alpha_i y_i\Big) \le \sum_{1\le i\le n} \alpha_i d(x_i, y_i)$$

for any (x_i) and (y_i) in M. One may argue that the choice of our convex combination in M depends on the retract r and the choice of the isometric embedding, the answer is yes. Therefore depending on the problem, one may have to be careful about this choice.
Let us introduce a notation which will be valide throughout this work: let M be a metric space and consider the natural embedding into $l^\infty(I)$ given by statement 1, set

$$M_\infty = co(M) \in \mathcal{A}\Big(l^\infty(I)\Big).$$

Clearly M_∞ is a hyperconvex subset of $l^\infty(I)$ and convex. The reason why we consider such set is because it is convex and bounded when M is bounded. We can also restrict the retract r to M_∞ into M.

3 Fixed point property in hyperconvex spaces

Sine and Soardi were the first to prove the following (although their proofs were given in different context)

Theorem 1. *Let H be a bounded hyperconvex metric space and $T : H \to H$ a nonexpansive map. Then $Fix(T)$ is not empty and is a hyperconvex.*

Note that the assumption *bounded* is essential. Indeed, it was unknown for a while whether bounded orbit will be enough to insure the existence of a fixed point.

Example. (Prus [6]) Let λ denote a Banach limit and define for each $(x_n) \in l^\infty$

$$T(x_1, x_2, ..) = (1 + \lambda((x_n)), x_1, x_2, ..) .$$

Then the orbit of 0 is bounded and T is fixed point free. Note that T is an isometry.

Remark. Since the assumption of boundedness seems essential, one may ask what happened if we only assume bounded orbit. Then using the natural embedding described above, one can prove [6] that the ε-fixed point set defined as

$$Fix_\varepsilon(T) = \{x \in H; d(x, Tx) \leq \varepsilon\}$$

is not empty and is hyperconvex.

Since $Fix(T)$ is hyperconvex, we know that there will be a nonexpansive retract from H into $Fix(T)$. But we do not have any precise information about such retract. Lin and Sine [9] did investigate this problem closely. In particular, they proved that there exists a retract which commutes with T.
A natural question to be asked is whether the conclusion of Theorem 1 still holds for any commutative family of nonexpansive mappings. This was answered by Baillon [2]. Indeed, Baillon proved that any decreasing family of nonempty bounded hyperconvex spaces has a nonempty intersection. The proof is very highly technical. From this, one can deduce the following

Theorem 2. *Let H be a bounded hyperconvex metric space and $\mathcal{F}$ a family of commutative nonexpansive self-maps of H. Then the common fixed point set $Fix(\mathcal{F})$ is not empty and is hyperconvex.*

4 KKM-maps and Ky Fan theorem

Among the results equivalent to the Brouwer's fixed point theorem, the theorem of Knaster-Kuratowski-Mazurkiewicz [3] occupies a special place. Let H be a metric space. The set of all subsets of H is denoted 2^H and in the linear case, the notation $conv(A)$ describe the

convex hull of A. A subset $A \subset H$ is called finitely closed if for every $x_1, x_2, .., x_n \in H$, the set $co(x_i) \cap A$ is closed. Note that $co(X)$ is always defined and belongs to $\mathcal{A}(H)$. If A is closed then obviously it is also finitely closed. Recall that a family $\{A_\alpha; A_\alpha \in 2^H\}$ is said to have the finite intersection property if the intersection of each finite subfamily is not empty.

Definition 3. Let H be a metric space and $X \subset H$. A multivalued mapping $G : X \to 2^X$ is called a Knaster-Kuratowski-Mazurkiewicz map (in short KKM-map) if

$$co(x_1, .., x_n) \subset \bigcup_{1 \leq i \leq n} G(x_i)$$

for any $x_1, .., x_n \in X$.

Theorem 3.(KKM-maps principle) *Let H be hyperconvex metric space, X an arbitrary subset of H, and $G : X \to 2^H$ a KKM-map such that each $G(x)$ is finitely closed. Then the family $\{G(x); x \in X\}$ has the finite intersection property.*

Proof. Assume not, i.e. there exists $x_1, ..x_n \in X$ such that $\bigcap G(x_i) = \emptyset$. Set $L = co(x_i)$ in H. Consider the hyperconvex set H_∞ and $C = conv(x_i)$ in H_∞. Let r be the nonexpansive retract $r : H_\infty \to H$ defined above. Note that $r(C) \subset L$. Our assumptions imply that $L \cap G(x_i)$ is closed for every $i = 1, 2, ..n$. Therefore, for every $c \in C$, there exists i_0 such that $r(c)$ does not belong to $L \cap G(x_{i_0})$ since $L \bigcap \cap G(x_i) = \emptyset$. Hence $d\Big(r(c), L \cap G(x_{i_0})\Big) > 0$ because $L \cap G(x_{i_0})$ is closed. Therefore, the function $\alpha(c) = \sum_i d\Big(r(c), L \bigcap \cap G(x_i)\Big)$ is not zero for any $c \in C$. Define the map $F : C \to C$ by

$$F(c) = \frac{1}{\alpha(c)} \sum_{1 \leq i \leq n} d\Big(r(c), L \bigcap \cap G(x_i)\Big) x_i \ .$$

Clearly, F is a continuous map. Using Brouwer's theorem, we get a fixed point c_0 of F, i.e. $F(c_0) = c_0$. Set

$$I = \Big\{i; d\Big(r(c_0), L \bigcap \cap G(x_i)\Big) \neq 0\Big\} \ .$$

Clearly we have

$$c_0 = \frac{1}{\alpha(c_0)} \sum_{1 \leq i \in I} d\Big(r(c_0), L \bigcap \cap G(x_i)\Big) x_i \ .$$

Therefore, $r(c_0) \notin \bigcup_{i \in I} G(x_i)$ and $r(c_0) \in co(\{x_i; i \in I\})$, contradicting the assumption

$$co\Big(\{x_i; i \in I\}\Big) \subset \bigcup_{i \in I} G(x_i) \ .$$

The proof is therefore complete.

As an immediate consequence, we obtain

Theorem 4. *Let H be a hyperconvex metric space and $X \subset H$ an arbitrary subset. Let $G : X \to 2^H$ be a KKM-map such that $G(x)$ is closed for any $x \in X$ and $G(x_0)$ is compact for some $x_0 \in X$. Then we have*

$$\bigcap_{x \in X} G(x) \neq \emptyset \ .$$

Notice that the compactness assumption of $G(x_0)$ may be a stronger one. We can still reach the conclusion if one involves an auxiliary multivalued map and a suitable topology on H (such as the ball topology for example).

Theorem 5. *Let H be a hyperconvex metric space and $X \subset H$ an arbitrary subset. Let $G : X \to 2^H$ be a KKM-map. Assume there is a multivalued map $K : X \to 2^H$ such that $G(x) \subset K(x)$ for every $x \in X$ and*

$$\bigcap_{x \in X} K(x) = \bigcap_{x \in X} G(x) \ .$$

If there is some topology on H such that each $K(x)$ is compact, then

$$\bigcap_{x \in X} G(x) \neq \emptyset \ .$$

The proof is obvious.

In order to prove the analogue of Ky Fan's fixed point result [3], we need the following lemma which is a direct application of KKM-maps [3].

Lemma. *Let H be a hyperconvex metric space and $X \in \mathcal{A}(H)$ compact. Let $F : X \to H$ be continuous. Then there exists $y_0 \in X$ such that*

$$d(y_0, F(y_0)) = \inf_{x \in X} d\Big(x, F(y_0)\Big) \ .$$

Proof. Consider the map $G : X \to 2^H$ defined by

$$G(x) = \Big\{ y \in X ; d(y, F(y)) \leq d(x, F(y)) \Big\} \ .$$

Since F is continuous, then $G(x)$ is closed for any $x \in X$. We claim that G is a KKM-map. Indeed, assume not. Then there exists $\{x_1, .., x_n\} \subset X$ and $y \in co(\{x_i\})$ such that $y \notin \bigcup_i G(x_i)$. This clearly implies

$$d(x_i, F(y)) < d(y, F(y)) \ , \quad \text{for } i = 1, .., n \ .$$

Let $\varepsilon > 0$ such that $d(x_i, F(y)) \leq d(y, F(y)) - \varepsilon$, for $i = 1, 2, .., n$. Hence $x_i \in B\Big(F(y), d(y, F(y)) - \varepsilon\Big)$, for $i = 1, .., n$. Therefore, we have $co(\{x_i\}) \subset B\Big(F(y), d(y, F(y)) - \varepsilon\Big)$, which implies $y \in B\Big(F(y), d(y, F(y)) - \varepsilon\Big)$. Clearly this gets us our contradiction which completes the proof of our claim.
By the compactness of X, we deduce that $G(x)$ is compact for any $x \in X$. Therefore, there exists $y_0 \in \bigcap_{x \in X} G(x)$. This clearly implies $d(y_0, F(y_0)) \leq d(x, F(y_0))$ for any $x \in X$ which implies

$$d(y_0, F(y_0)) = \inf_{x \in X} d\Big(x, F(y_0)\Big)$$

and the proof is complete.

We now are ready to state Ky fan's fixed point theorem [3] in hyperconvex metric spaces.

Theorem 6. *Let H be a hyperconvex metric space and $X \in \mathcal{A}(H)$ compact. Let $F : X \to H$ be continuous and such that, for every $c \in X$, with $c \neq F(c)$, there exsits $\alpha \in (0,1)$ such that*

$$(*) \quad X \cap B\Big(c, \alpha d(c, F(c))\Big) \cap B\Big(F(c), (1-\alpha)d(c, F(c))\Big) \neq \emptyset \ .$$

Then F has a fixed point, i.e. $F(y) = y$ for some $y \in X$.

Note that the condition $(*)$ means that a metric convex combination of c and $F(c)$ belongs to X. In particular, it is satisfied if $F(X) \subset X$.

Proof. By the previous lemma, there exists $y_0 \in X$ such that

$$d(y_0, F(y_0)) = \inf_{x \in X} d\Big(x, F(y_0)\Big) \ .$$

We claim that such element y_0 is a fixed point of F. Indeed, assume not, i.e. $y_0 \neq F(y_0)$. Then our assumption on X implies the existence of $\alpha \in (0,1)$ such that

$$X \cap B\Big(c, \alpha d(c, F(c))\Big) \cap B\Big(F(c), (1-\alpha)d(c, F(c))\Big) \neq \emptyset \ .$$

Let $x \in X \cap B\Big(y_0, \alpha d(y_0, F(y_0))\Big) \cap B\Big(F(y_0), (1-\alpha)d(y_0, F(y_0))\Big)$. Then $d(x, F(y_0) = (1-\alpha)d(y_0, F(y_0))$. Since $d(y_0, F(y_0)) \leq d(x, F(y_0))$, we clearly get a contradiction. This completes our proof.

This theorem implies that any continuous self-map of a compact admissible subset of a hyperconvex metric space has a fixed point. This can be easily extended to a class of subsets other than admissible ones, i.e. we may relax this assumption by a kind of metric convexity.

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