# Bridging Proportional Reasoning and Algebraic Reasoning:

## A Focus on Co-variation and Invariance Using Contextualized Problems


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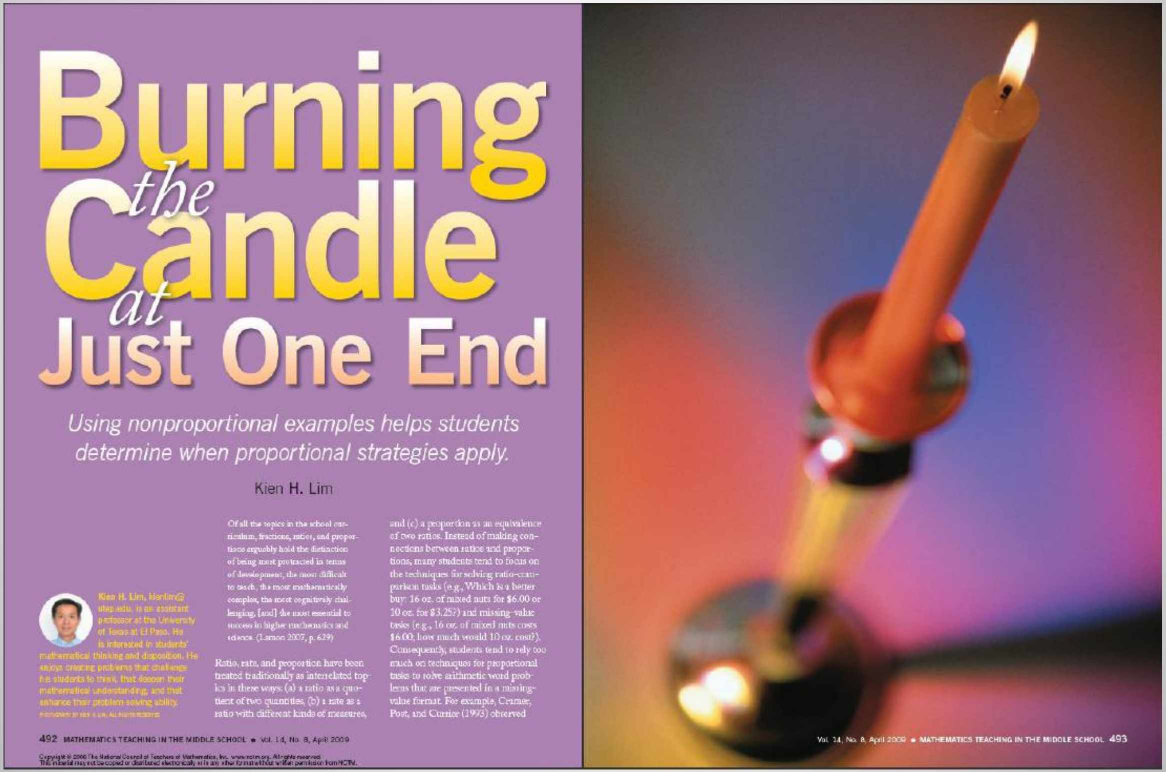
All the contextualized problems in this presentation are from this article in



# Burning the Candle at Just One End

*Using nonproportional examples helps students determine when proportional strategies apply.*

Kien H. Lim

Of all the topics in the school curriculum, fractions, ratios, and proportions arguably hold the distinction of being most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, [and] the most essential to success in higher mathematics and science. (Lamon 2007, p. 629)

Ratio, rate, and proportion have been treated traditionally as interrelated topics in these ways: (a) a ratio as a quotient of two quantities, (b) a rate as a ratio with different kinds of measures, and (c) a proportion as an equivalence of two ratios. Instead of making connections between ratios and proportions, many students tend to focus on the techniques for solving ratio-comparison tasks (e.g., Which is a better buy: 16 oz. of mixed nuts for $6.00 or 10 oz. for $3.25?) and missing-value tasks (e.g., 16 oz. of mixed nuts costs $6.00, how much would 10 oz. cost?). Consequently, students tend to rely too much on techniques for proportional tasks to solve arithmetic word problems that are presented in a missing-value format. For example, Cramer, Post, and Currier (1993) observed

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# Outline of Presentation

- Proportional Reasoning & Beginning Algebra
- Connecting Proportion to Algebra
- Pedagogical Suggestions
  - Focus on Co-variation and Invariance
  - Use of Non-proportional Situations

# Proportional Reasoning

- ## Missing-value problems

Two different candles, P and Q, lighted at the same time were burning at different, but constant, rates. When candle P had burned 16 mm, candle Q had burned 10 mm. When candle Q had burned 35 mm, how many mm would candle P have burned?

$$\frac{16}{10} = \frac{x}{35}$$

$$10x = 16 \cdot 35$$

$$\frac{10x}{10} = \frac{560}{10}$$

$$\boxed{x = 56\text{mm}}$$

- ## Strategy 1: Setting up a proportion

Is this student reasoning proportionally?

How do you think the student will solve this problem?

Two identical candles, A and B, lighted at different times were the same constant rate. When candle A had burned 20 mm, ca had burned 12 mm. When candle B had burned 30 mm, how m would candle A have burned?

$$\frac{20}{x} = \frac{12}{30}$$

$$12x = 20 \cdot 30$$

$$\frac{12x}{12} = \frac{600}{12}$$

$$x = 50\text{mm}$$

# Proportional Reasoning

- Missing-value problems

Two different candles, P and Q, lighted at the same time were burning at different, but constant, rates. When candle P had burned 16 mm, candle Q had burned 10 mm. When candle Q had burned 35 mm, how many mm would candle P have burned?

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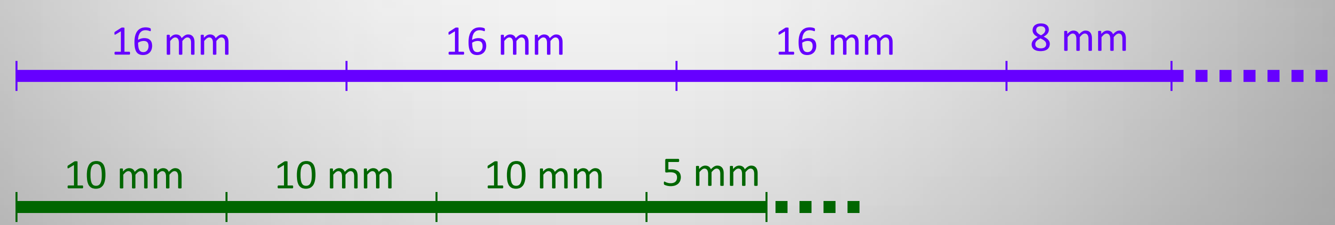
- Strategy 1: Setting up a proportion

- Strategy 2: Coordinating quantities




Which strategy demonstrates proportional reasoning?

# Beginning Algebra

- Solving Linear Equations
  - One-step: $3x = 20$ or $x + 8 = 20$
  - Two-step: $3x + 8 = 20$
  - Multi-step: $3x + 8 = 4(x - 3)$

- Representing and Understanding Linear Functions
  - Equation form (e.g., $y = 3x + 8$)
    Graphical form
    Numerical form
  - Slope
    $y$-intercept

# Proportion-Algebra Connection

Proportion

$$\frac{a}{b} = \frac{x}{d}$$

Beginning Algebra

$$y = mx + b$$

Co-variation

&

Invariance

$$\boxed{y = mx}$$

# Pedagogical Suggestions

- Build on Proportional Missing-value Problems
  - ✓ Identifying quantities that change (i.e. variables)

Two different candles, P and Q, lighted at the same time were burning at different, but constant, rates.  When candle P had burned 16 mm, candle Q had burned 10 mm. When candle Q had burned 35 mm, how many mm would candle P have burned?

(a) Identify the quantities in this problem.

16 mm      10 mm

?      35 mm

These are numbers, not quantities!

tities,

The length burned for candle P at the first moment. (known)

The length burned for candle P at the second moment. (unknown)

The length burned for candle Q at the first moment. (known)

The length burned for candle Q at the second moment. (known)

# Pedagogical Suggestions

- Build on Proportional Missing-value Problems
  - ✓ Identifying quantities that change (i.e. variables) and how those quantities are related
  - ✓ Focusing on Co-variation and Invariance

Two different candles, P and Q, lighted at the same time were burning at different, but constant, rates. When candle P had burned 16 mm, candle Q had burned 10 mm. When candle Q had burned 35 mm, how many mm would candle P have burned?

(a) Identify the quantities in this problem.

(b) Let $p$ represent the number of mm that candle P had burned when candle Q had burned $q$ mm. Write an equation to relate $p$ and $q$.

Students need to mentally act out the problem situation.

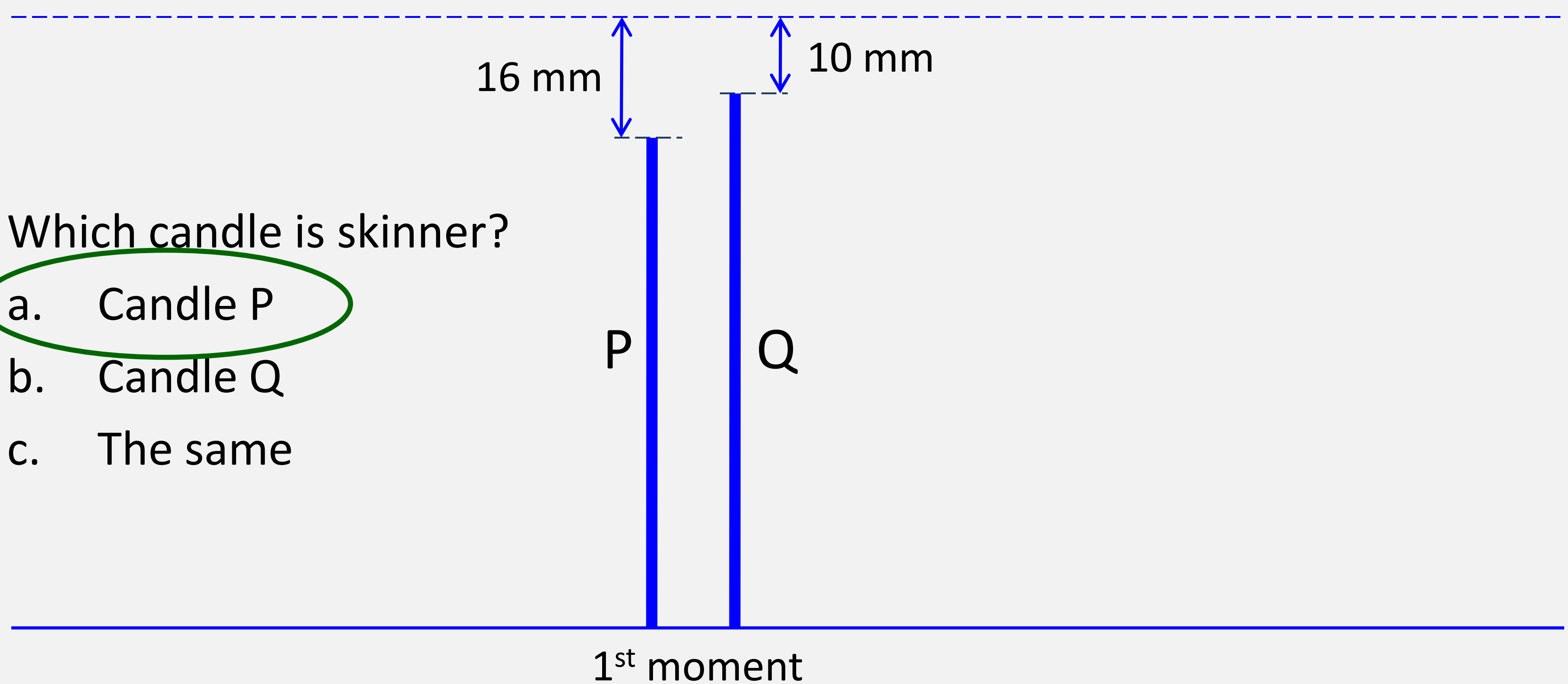
Encourage them to draw diagrams.

Two different candles, P and Q, lighted at the same time were burning at different, but constant, rates.  When candle P had burned 16 mm, candle Q had burned 10 mm. When candle Q had burned 35 mm, how many mm would candle P have burned?

Whenever appropriate, pose questions to make students think.

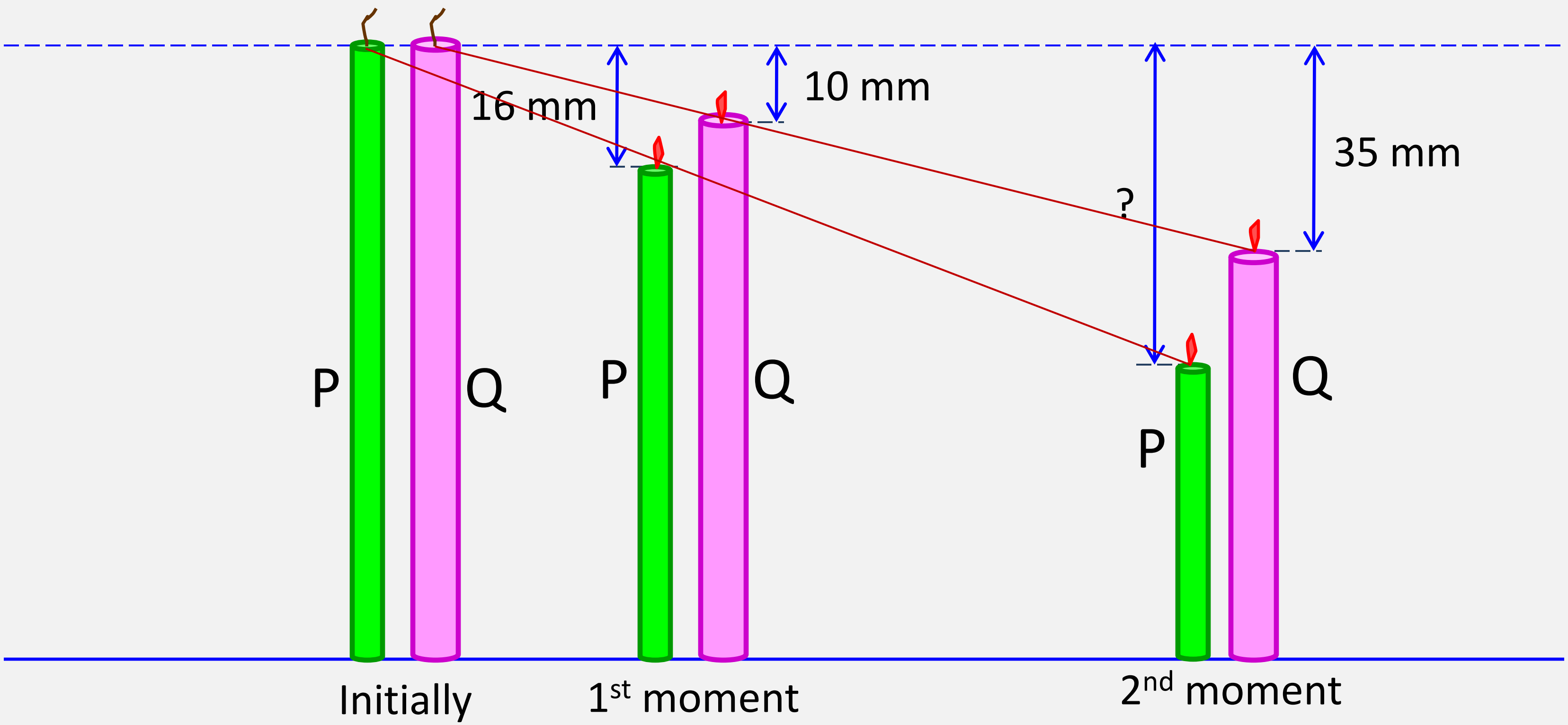
Which candle is skinner?

a. Candle P
b. Candle Q
c. The same



Two different candles, P and Q, lighted at the same time were burning at different, but constant, rates. When candle P had burned 16 mm, candle Q had burned 10 mm. When candle Q had burned 35 mm, how many mm would candle P have burned?

What is invariant in this problem? The burning rate of each candle.



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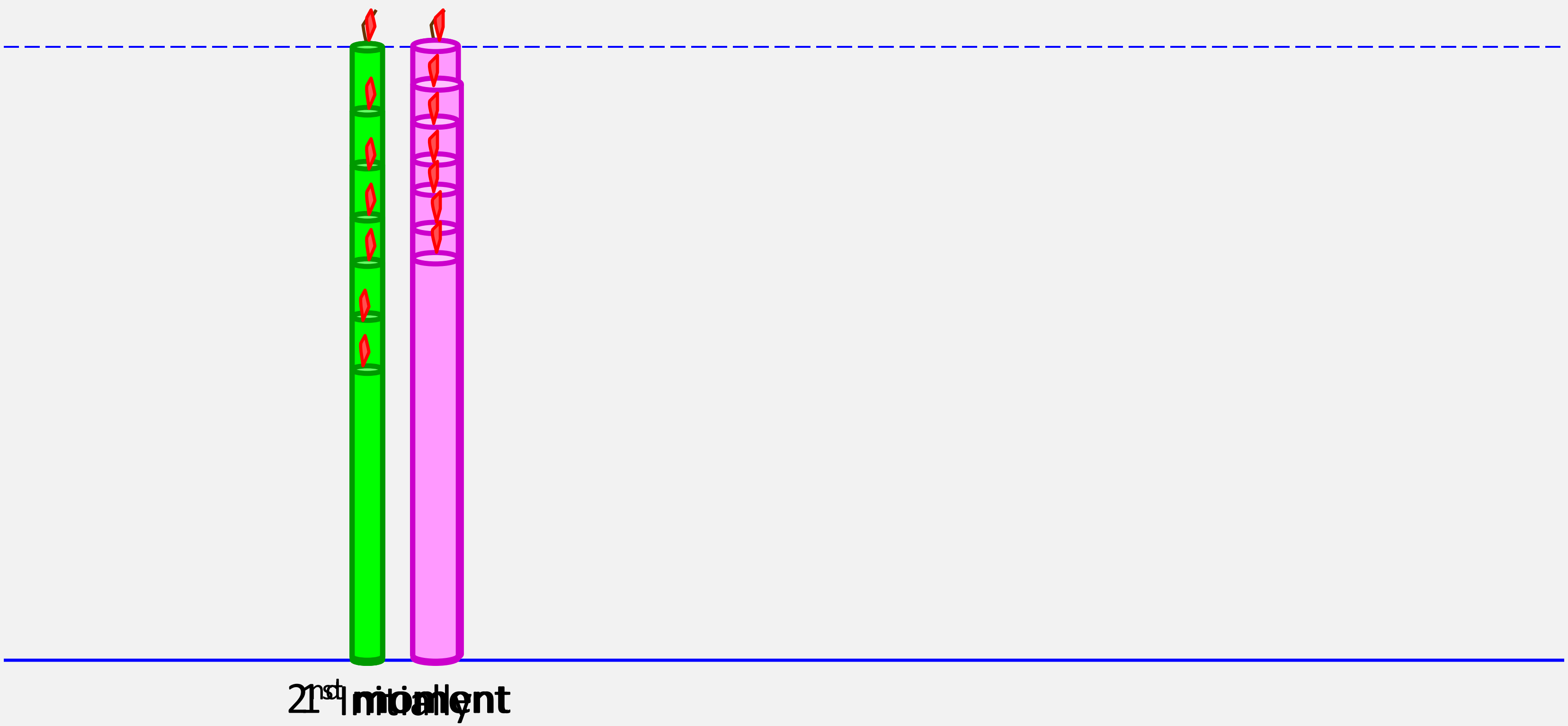

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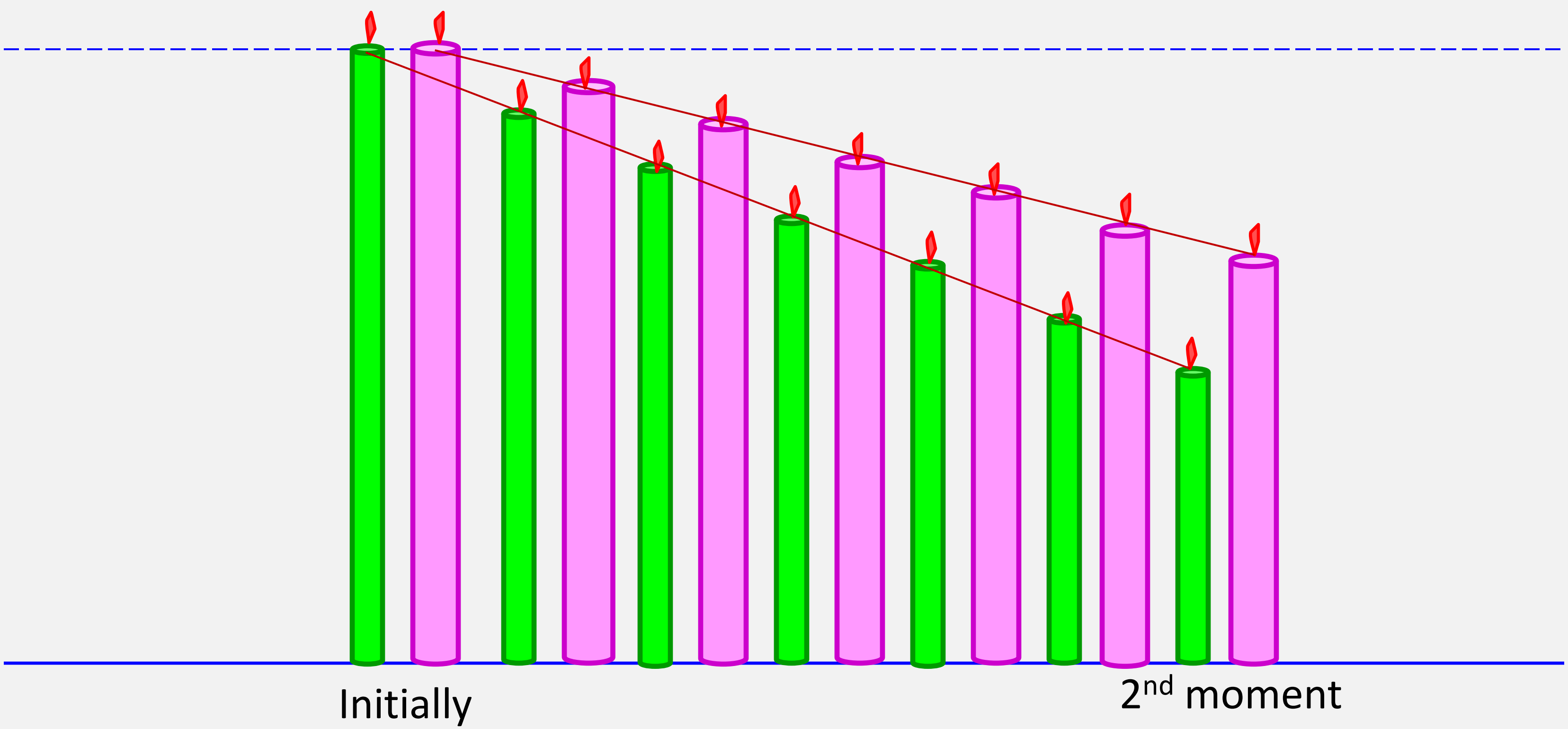

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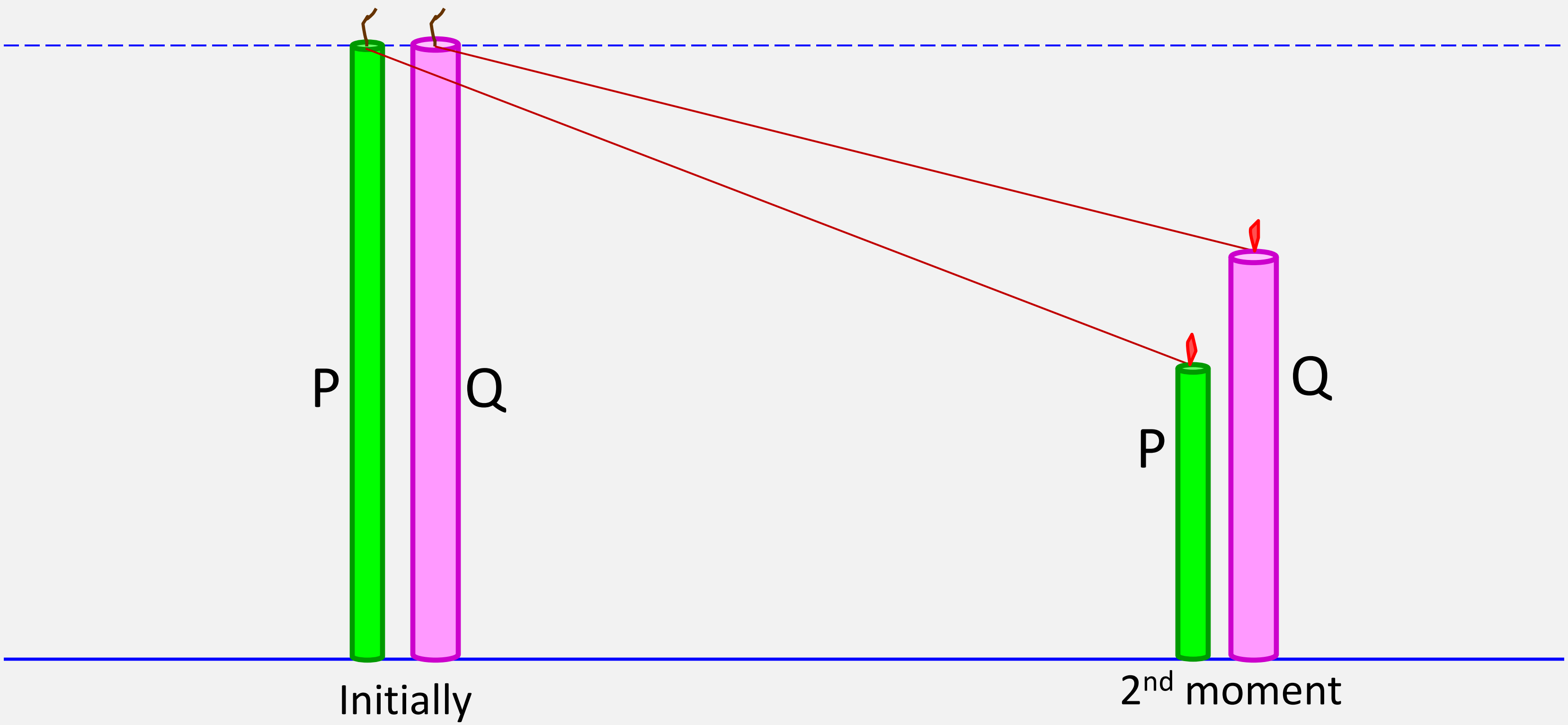
What else is invariant in this problem?

The ratio of 16/10 is invariant.

What does the ratio 16/10, or the value 1.6, represent?



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What else is invariant in this problem?

The ratio of 16/10 is invariant.

What does the ratio 16/10, or the value 1.6, represent?

Candle P burned 1.6 mm for every 1mm burned by Candle Q.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Length (mm) Burned by Candle P | 0 | 1.6 | 3.2 | 4.8 | 6.4 | 8 | 16 | 32 | 48 | $p$ |
| Length (mm) Burned by Candle Q | 0 | 1 | 2 | 3 | 4 | 5 | 10 | 20 | 30 | $q$ |

(b) Let $p$ represent the number of mm that candle P had burned when candle Q had burned $q$ mm. Write an equation to relate $p$ and $q$.

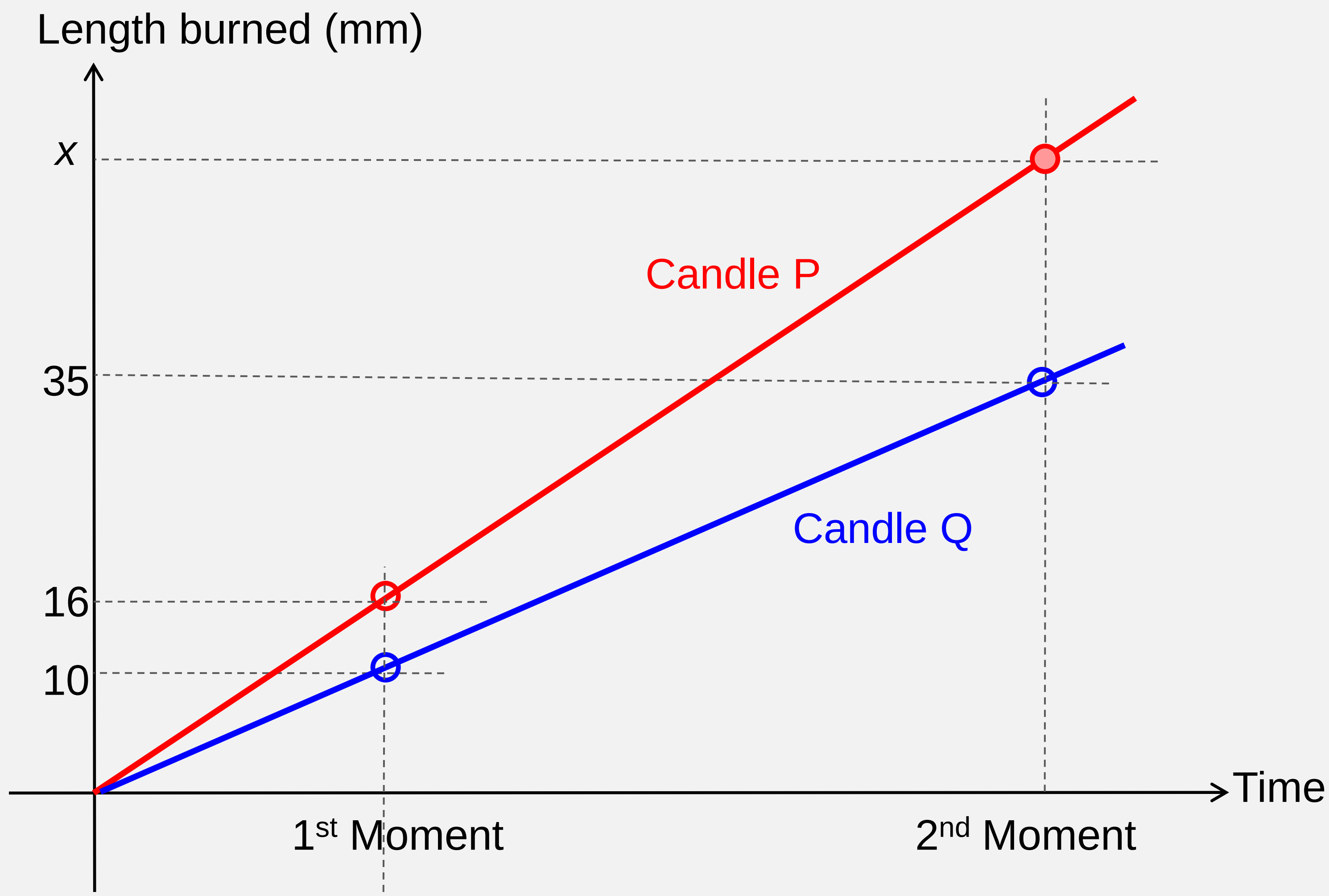
$$q = 1.6p$$

# Pedagogical Suggestions

- Build on Proportional Missing-value Problems
  - ✓ Focusing on co-variation and invariance
  - ✓ Making connections among various representations

Two different candles, P and Q, lighted at the same time were burning at different, but constant, rates. When candle P had burned 16 mm, candle Q had burned 10 mm. When candle Q had burned 35 mm, how many mm would candle P have burned?

(c) How else can we show the relationship between the variables?



# Pedagogical Suggestions

- Build on Proportional Missing-value Problems
  - ✓ Focusing on co-variation and invariance
  - ✓ Making connections among various representations
  - ✓ Interpreting slope meaningfully

What does the slope represent of each line represent?

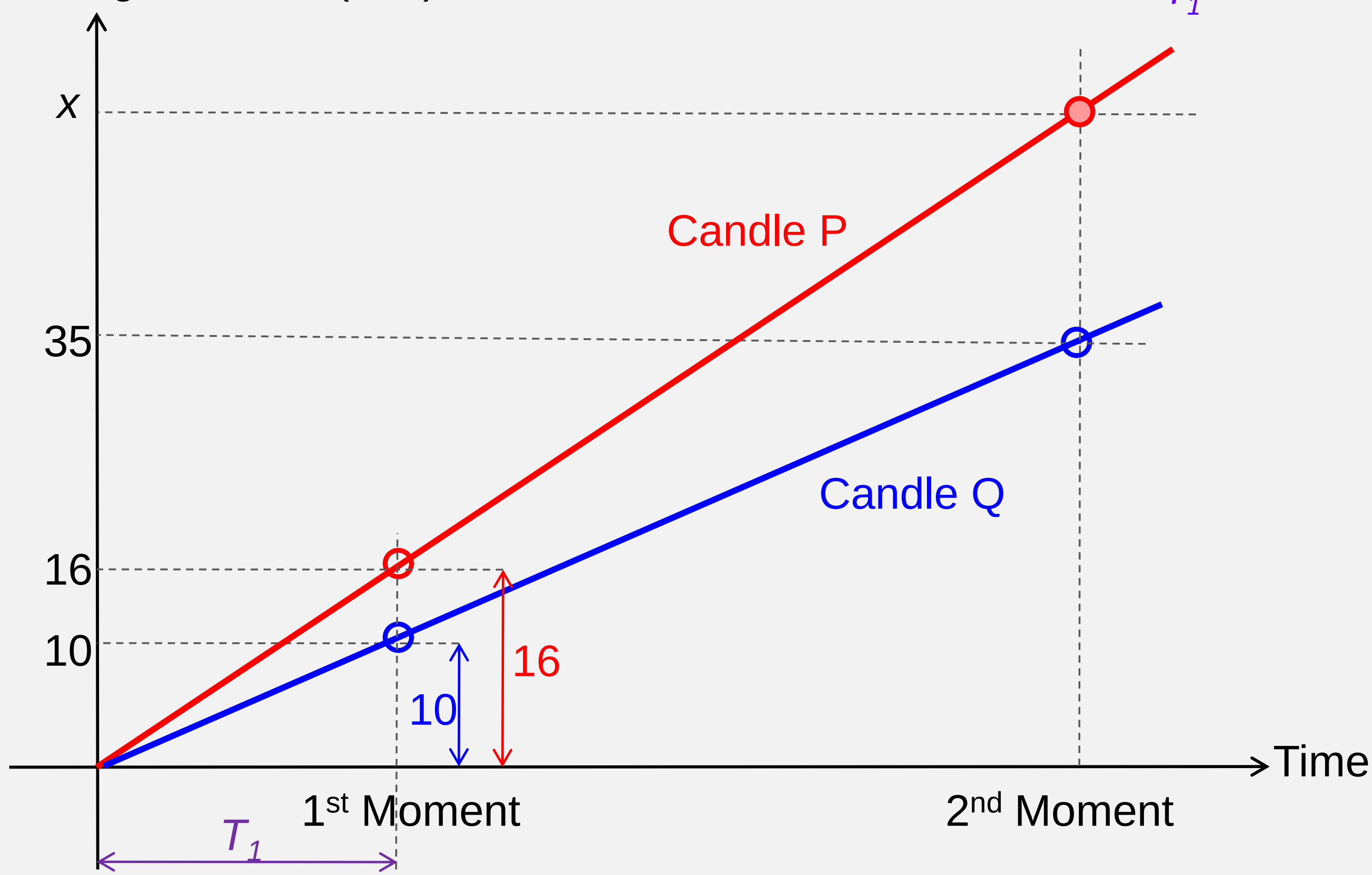
The burning rate for each candle (value is unknown). $\frac{16}{T_1}$ & $\frac{10}{T_1}$

How are the slopes related?

Candle P burned 1.6 times faster than Candle Q.

Slope of line for Candle P is 1.6 times that of Candle Q.

$$\frac{\frac{16}{T_1}}{\frac{10}{T_1}} = \frac{16}{10} = \frac{1.6}{1}$$



# Pedagogical Suggestions

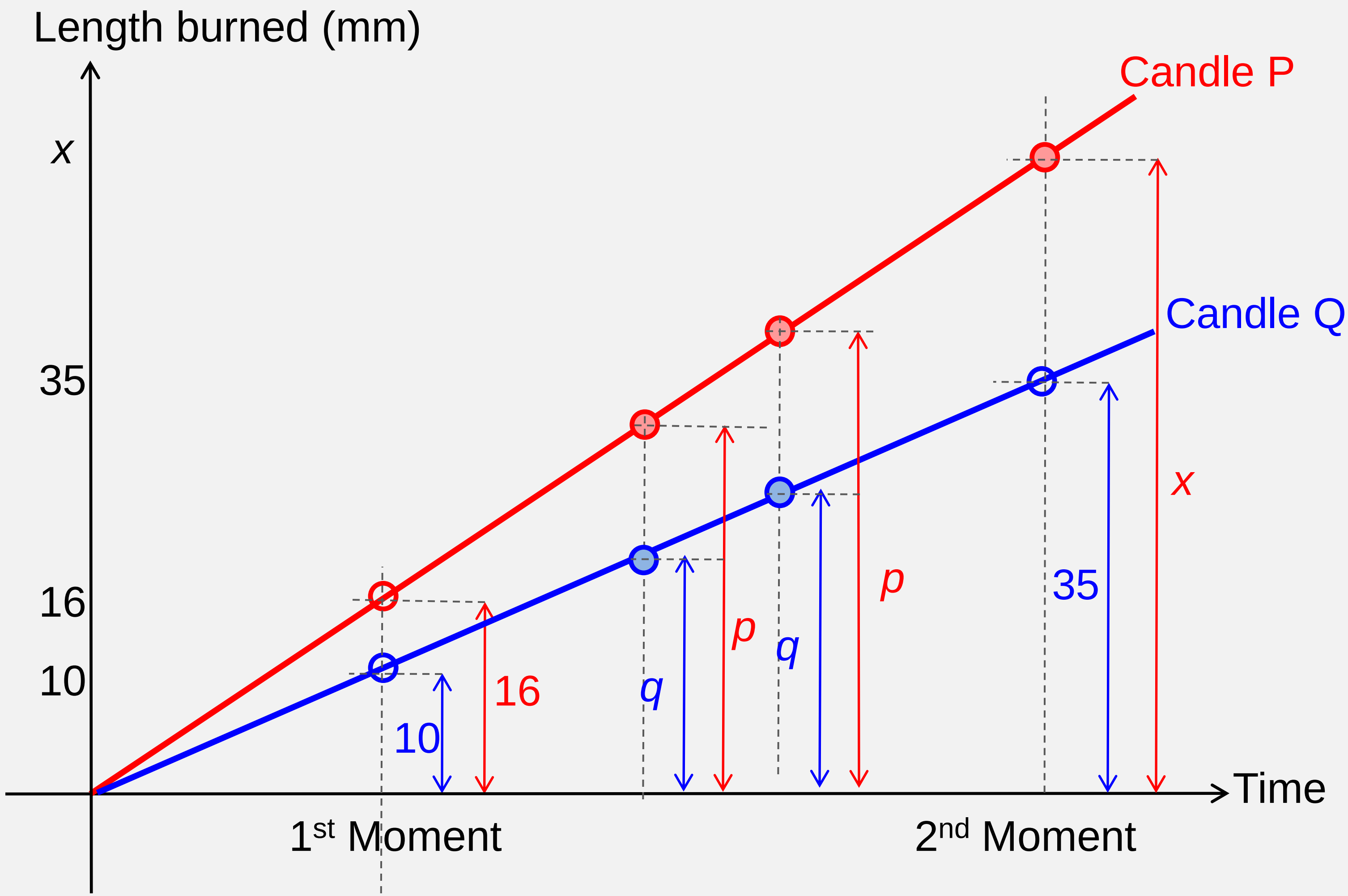
- Build on Proportional Missing-value Problems
  - ✓ Focusing on co-variation and invariance
  - ✓ Making connections among various representations
  - ✓ Interpreting slope meaningfully
  - ✓ Recognizing that ratio is invariant

$$\frac{16}{10} = \frac{35}{x} \qquad \frac{16}{10} = \frac{p}{q}$$

$$\frac{16}{10} = \frac{35}{x}$$

$$p = 1.6q$$

$$\boxed{\frac{16}{10} = \frac{1.6}{1} = \frac{x}{35} = \frac{p}{q}}$$



# Pedagogical Suggestions

- Build on Proportional Missing-value Problems
  - ✓ Focusing on co-variation and invariance
  - ✓ Making connections among various representations
  - ✓ Interpreting slope meaningfully
  - ✓ Recognizing that ratio is invariant

- Include Non-proportional Missing-value Problems

"Part of understanding [proportionality] is knowing what it is *not* and when it does *not* apply."

(Lamon, 2007, p. 647)

# Pedagogical Suggestions

Two identical candles, A and B, lighted at different times were burning at the same constant rate. When candle A had burned 20 mm, candle B had burned 12 mm. When candle B had burned 30 mm, how many mm would candle A have burned?

$$30 - x = \overbrace{20 - 12}^{\text{8 mm difference}}$$

$$y - x = 8$$

- Include Non-proportional Missing-value Problems
  - ✓ Invariant Difference (i.e., a linear function with a slope of 1)

# Pedagogical Suggestions

A candle is burning at a constant rate.
When it has burned 30 mm, its height is 75 mm.
When it has burned 60 mm, what is the candle's height?

$$60 + x \;=\; \overbrace{30 + 75}^{\text{Original height of 105 mm}}$$

$$y + x \;=\; 105$$

- Include Non-proportional Missing-value Problems
  - ✓ Invariant Difference
  - ✓ Invariant Sum (i.e., a linear function with a slope of -1)

# Pedagogical Suggestions

An altar in a church needs to be lighted, one special candle at a time continuously for a week-long festival. If the church uses special candles that last 7 hours each, then the church needs 24 such candles. If the church uses special candles that last 8 hours each, how many such candles will the church need?

$$x \cdot 8 = 24 \cdot 7 \qquad \longrightarrow \qquad xy = 168$$

- Include Non-proportional Missing-value Problems
  - ✓ Invariant Difference
  - ✓ Invariant Sum
  - ✓ Invariant Product (i.e., a reciprocal function)

# Pedagogical Suggestions

- Build on Proportional Missing-value Problems
  - ✓ Focusing on co-variation and invariance
  - ✓ Making connections among various representations
  - ✓ Interpreting slope meaningfully
  - ✓ Recognizing that ratio is invariant

- Include Non-proportional Missing-value Problems
  - ✓ Invariant Difference
  - ✓ Invariant Sum
  - ✓ Invariant Product

# Types of Non-proportional Missing-value Problems

- Invariant Difference
- Invariant Sum
- Invariant Product
- Invariant Quantity

A group of 5 musicians plays a piece of music in 10 minutes. Another group of 35 musicians will play the same piece of music. How long will it take this group to play it?

$\frac{5}{10} = .5 = \frac{1}{2}$

Cross multiply to figure out since it is proportional

$\frac{5}{10} = \frac{35}{X}$

$5x = 350$

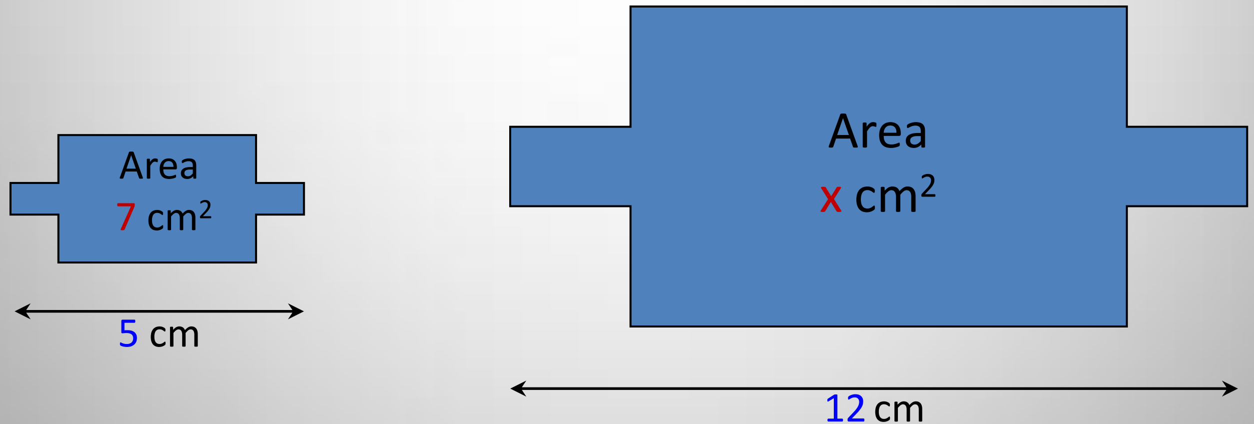
$x = \frac{350}{5}$

$x = 70$ minutes or $\boxed{1\text{ hr } 10\text{ min}}$

# Types of Non-proportional Missing-value Problems

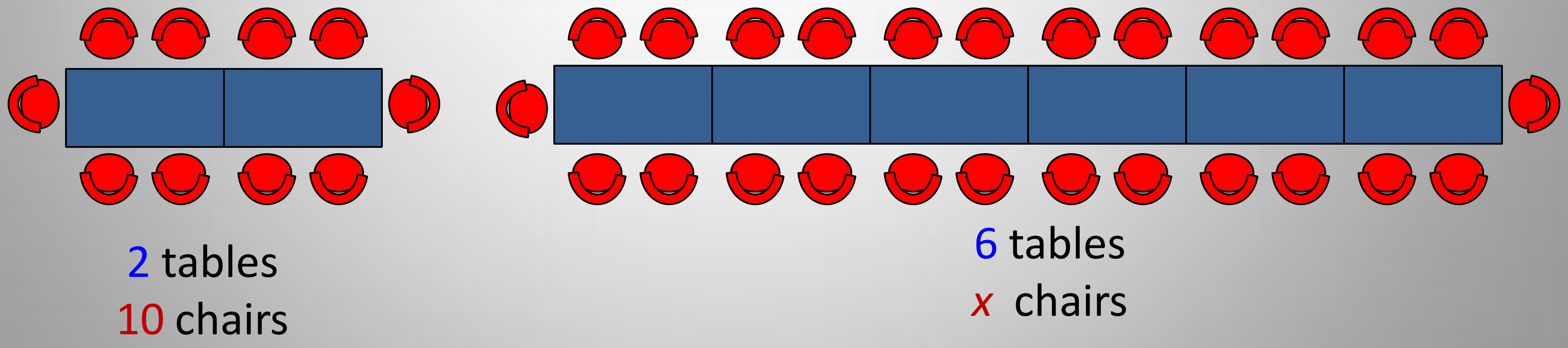
- Invariant Difference
- Invariant Sum
- Invariant Product
- Invariant Quantity
- Invariant Shape (e.g. enlargement involving area)

$$\left(\frac{12}{5}\right)^2 = \frac{x}{7}$$



# Types of Non-proportional Missing-value Problems

- Invariant Difference
- Invariant Sum
- Invariant Product
- Invariant Quantity
- Invariant Shape (e.g. enlargement involving area)
- Affine Function ($y = mx + b$)



# Types of Non-proportional Missing-value Problems

- Invariant Difference
- Invariant Sum
- Invariant Product
- Invariant Quantity
- Invariant Shape (e.g. enlargement involving area)
- Affine Function ($y = mx + b$)
- Other Non-linear Function

If the number of bacteria multiplies by a factor of 4 every 10 minutes, then the number of bacteria will multiply by a factor of __ every 20 minutes.