Mathematical Habits of Mind: An Organizing Principle for Curriculum Design

Al Cuoco

Center for Mathematics Education

EDC

acuoco@edc.org

Slides available at `http://www2.edc.org/CME/showcase/`

What mathematicians most wanted and needed from me was to learn my ways of thinking, and not in fact to learn my proof of the geometrization conjecture for Haken manifolds.

—William Thurston

On Proof and Progress in Mathematics

Surprise 1

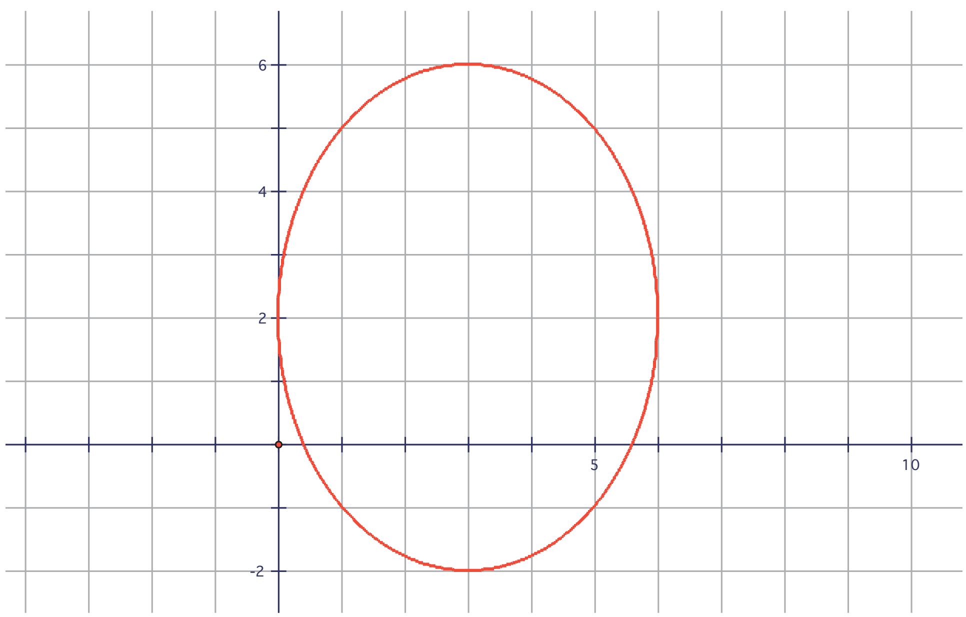
Graph:

$$16x^2 - 96x + 9y^2 - 36y + 36 = 0$$

Solution:

$$16x^2 - 96x + 9y^2 - 36y + 36 = 0 \quad \Rightarrow$$

$$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{16} = 1 \qquad \Rightarrow$$



Surprise 2

Consider the system of linear equations

$$\begin{aligned} 3x + 4y - 2z + w &= 6 \\ 5x - 2y + 2z + 2w &= 7 \\ -3x + 2y - z + 5w &= 3 \end{aligned}$$

1. Find a solution to this system.

2. Suppose you have two solutions to this system, say

   $A = (r_1, r_2, r_3, r_4)$ and $B = (s_1, s_2, s_3, s_4)$.

   Is $A + B$ a solution to the system? Why or why not?

3. Write down a system of equations that has the property that if A is a solution, so is $3A$.

Today's Theme:

Helping Students Develop Mathematical Habits of Mind without Compromising Key Concepts in the Syllabus

Some Ideas

1. Make mathematical thinking a key concept in the syllabus

2. Be explicit about one's own thinking

3. Work on problems *with* students

4. Make thought experiments an integral part of the experience

- *visualize geometric phenomena*

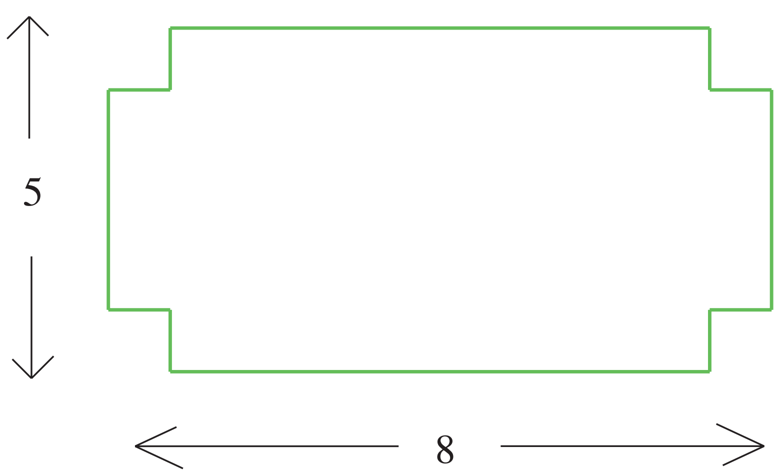
  $*$ solids of revolution

- *picture calculations*

  $*$ $(x-1)(x^4+x^3+x^2+x+1)$

- *imagine change*

  $*$



5. Experience before formality

> *. . . a perennial problem with university math lectures, where there is typically almost zero time delay between the example/process and the theorem/concept, and no clear message that student activity/thought is needed to cement/clarify the transition.*
>
> —Tony Gardiner

6. Look to the mathematics itself

> Traditional branches of mathematics—analysis, algebra, arithmetic, geometry, and topology—are not only compartments for certain kinds of results, they are also descriptors for the habits of mind indigenous to that discipline.

Algebraic Habits

- Abstracting regularity from calculations

n	$f(n)$	Δ	Δ^2	Δ^3	Δ^4
0	1	-2	14	12	0
1	-1	12	26	12	0
2	11	38	38	12	0
3	49	76	50	12	0
4	125	126	62	12	
5	251	188	74		
6	439	262			
7	701				

- Seeking structural similarities

Find a polynomial that agrees with

Input	Output
0	-5
1	-6
2	7
3	166
4	843
5	2770

In general, we're looking for a polynomial f such that

$$\begin{aligned} f(a_1) &= b_1 \\ f(a_2) &= b_2 \\ \vdots \quad & \quad \vdots \\ f(a_r) &= b_r \end{aligned}$$

By the "remainder theorem," another way

to say this is that we want f to satisfy

$$\begin{aligned}
f(x) &= (x-a_1)q_1(x)+b_1 \\
f(x) &= (x-a_2)q_2(x)+b_2 \\
\vdots & \qquad\qquad \vdots \\
f(x) &= (x-a_r)q_r(x)+b_r
\end{aligned}$$

This is similar in structure to the arithmetic puzzle:

> "The reminder when my age is divided by 3 is 2. When I divide it by 5, the remainder is 3. When I divide it by 7, the remainder is 1. How old am I?"

We want a number y so that

$$\begin{aligned}
y &= 3q_1+2 \\
y &= 5q_2+3 \\
y &= 7q_3+1
\end{aligned}$$

In both cases, we are working in an algebraic system that supports "long division," and we are looking for an object that leaves prescribed remainders when divided by a set of divisors.

Or, we are looking for a solution to a simultaneous set of *congruences*:

$$\left.\begin{array}{lll} f(x) & \equiv b_1 & \pmod{(x-a_1)} \\ f(x) & \equiv b_2 & \pmod{(x-a_2)} \\ \vdots & & \vdots \\ f(x) & \equiv b_r & \pmod{(x-a_r)} \end{array}\right\}$$

or

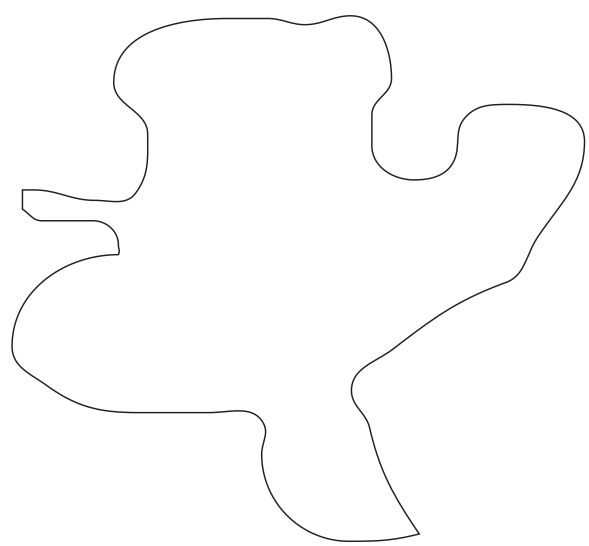
$$\left\{\begin{array}{lll} y & \equiv b_1 & \pmod{a_1} \\ y & \equiv b_2 & \pmod{a_2} \\ \vdots & & \vdots \\ y & \equiv b_r & \pmod{a_r} \end{array}\right.$$

This "sameness" can be made precise and leads to a deep structural similarity between $\mathbf{Z}$ and $\mathbf{Q}[\mathbf{x}]$.

- extension
- decomposition
- representation
- localization
- reduction,
- ...

Analytic Habits

- Reasoning by continuity
  - Was there a time in your life when your height in inches was equal to your weight in pounds? (T. Banchoff)
  - Is there a line that cuts the area of this region in half?



- Extension by continuity
  - What are reasonable definitions for
    - 2^0
    - 2^{-3}
    - $2^{\frac{-3}{2}}$
    - $2^{\sqrt{2}}$

- looking at extreme cases
- using approximation
- passing to the limit
- ...

About "pattern sniffing"

The question of how we deal with patterns is an important one. When it is unaccompanied by some form of a theory, simple pattern recognition lacks real value. But what exactly is a "theory" for a beginning learner? Patterns need to have "meaning". Experience of "patterns" needs always to be embedded in a "context" and to be accompanied by some form of "interpretation". I would say that "meaning", "context", "interpretation" are rudimentary forms of "theory". I believe there are lots of "theories" that even inexperienced learners carry around with them and use to determine whether or not they believe in the patterns they see.

—Glenn Stevens

Why Habits of Mind?

For *all* students, whether they eventually build houses, run businesses, use spreadsheets, or prove theorems, the real utility of mathematics is not that you can use it to figure the slope of a wheelchair ramp, but that it provides you with the intellectual schemata necessary to make sense of a world in which the products of mathematical thinking are increasingly pervasive in almost every walk of life.

This is not to say that other facets of mathematics should be neglected; questions of content, applications, cultural significance, and connections are all essential in the design of a mathematics program. But without explicit attention to mathematical ways of thinking, the goals of "intellectual sophistication" and "higher order thinking skills" will remain elusive.