What is Mathematics?

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Mathematics teaching must not appeal to gimmicks, entertainment, or contingencies of reward and punishment, but focus solely on the learner’s *intellectual need* by fully utilizing humans’ remarkable capacity to be puzzled.

Nor should mathematics curricula compromise the *mathematical integrity* of their contents. The mathematical integrity of a curricular content is determined by the *ways of understanding* and *ways of thinking* that have evolved in many centuries of mathematical practice and continue to be the ground for scientific advances.

An important goal of research in mathematics education is, therefore, to identify these *ways of understanding* and *ways of thinking* and recognize, when possible, their development with learners and in the history of mathematics, and, accordingly, develop and implement mathematics curricula that position the mathematical *integrity of the content* taught and *intellectual need* of the student at the center of the instructional effort.

What is mathematics?

1. What is the ***mathematics*** that mathematicians practice?
2. What is the ***mathematics*** that mathematicians value and perceive to know?
3. What is the ***mathematics*** that we aim for our students to know?

   “The goal of instruction should be unambiguous: to gradually refine and advance students’ current knowledge of mathematics toward ***contemporary mathematics***.”

4. What is ***contemporary mathematics***?
5. What are the essential differences among ***contemporary mathematics, 19th century mathematics, Renaissance mathematics,*** and ***Greek mathematics***?
6. Are, if so how are, these differences relevant to ***mathematics*** learning and teaching?
7. What is the ***mathematics*** that teachers at a particular grade level need to know to be effective?
8. What is the ***mathematics*** that we currently teach?
9. Is that the ***mathematics*** that we should teach?

A survey of mathematics teachers

Question 1:

- Why do we teach the *long division algorithm,* the *quadratic formula, techniques of integration,* and so on when one can perform arithmetic operations, solve many complicated equations, and integrate complex functions quickly and accurately using electronic technologies?

Teachers' Answers	Justifications
“These materials appear on standardized tests.”	Teachers must prepare students for tests mandated by their districts.
“One should be able to solve problems independently in case a suitable calculator is not present.”	Teachers must teach students to do calculations without a calculator, especially those needed in daily life.
“Such topics are needed to solve real-world problems and to learn more advanced topics.”	Teachers must prepare students for more advanced courses where certain computational skills might be assumed.

Teachers' answers are external to mathematics as a discipline

The justifications for these answers are:

- **neither cognitive**
  - Role of computational skills in *one's conceptual development of mathematics*
- **nor epistemological**
  - Role of computations in the development of mathematics
- **mainly social**
  - Role of computational skills in the context of social expectations

Question 2:

- Why teach proofs?

Typical Answer:

- So that students can be *certain* that the theorems we teach them are true.

While this is an adequate answer—both cognitively and (by inference) epistemologically—it is incomplete.

The teachers had little to say when skeptically confronted about their answers by being asked:

- Do you or your students doubt the truth of theorems that appear in textbooks?
- Is certainty the only goal of proofs?
- The theorems in Euclidean geometry, for example, have been proven and re-proven for millennia. We are certain of their truth, so why do we continue to prove them again and again?

- Textbooks and teachers at all levels tend to view mathematics
  - in terms of subject matter such as definitions, theorems, proofs, algorithms, and problems and their solutions.
  - not in terms of conceptual tools necessary to construct such entities.
- Knowledge of subject matter is indispensable, but it is not sufficient.
- Conceptual tools constitute an important category of knowledge different from the subject matter category.

What exactly are these two categories of knowledge, “subject matter” and “conceptual tools”?

What is the basis for the argument that both categories are needed?

Proving*, *Proof*, *Proof Scheme

- ***Proving*** is the mental act employed by a person to remove doubts about the truth of an assertion.
- Processes of proving:
  - ***Ascertaining***: what an individual employs to remove her or his doubts.
  - ***Persuading***: what an individual employs to remove others' doubts.

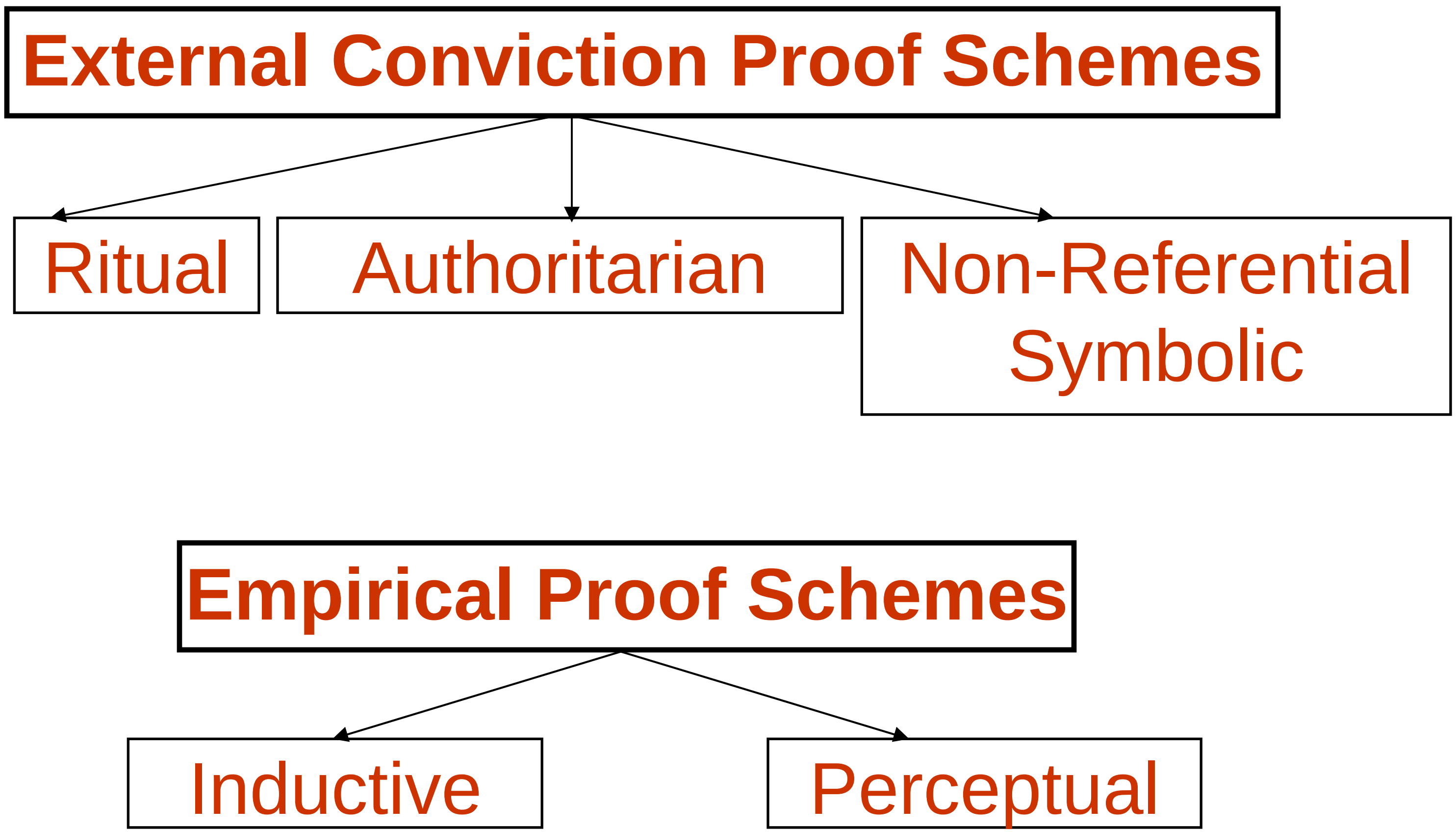
Proof

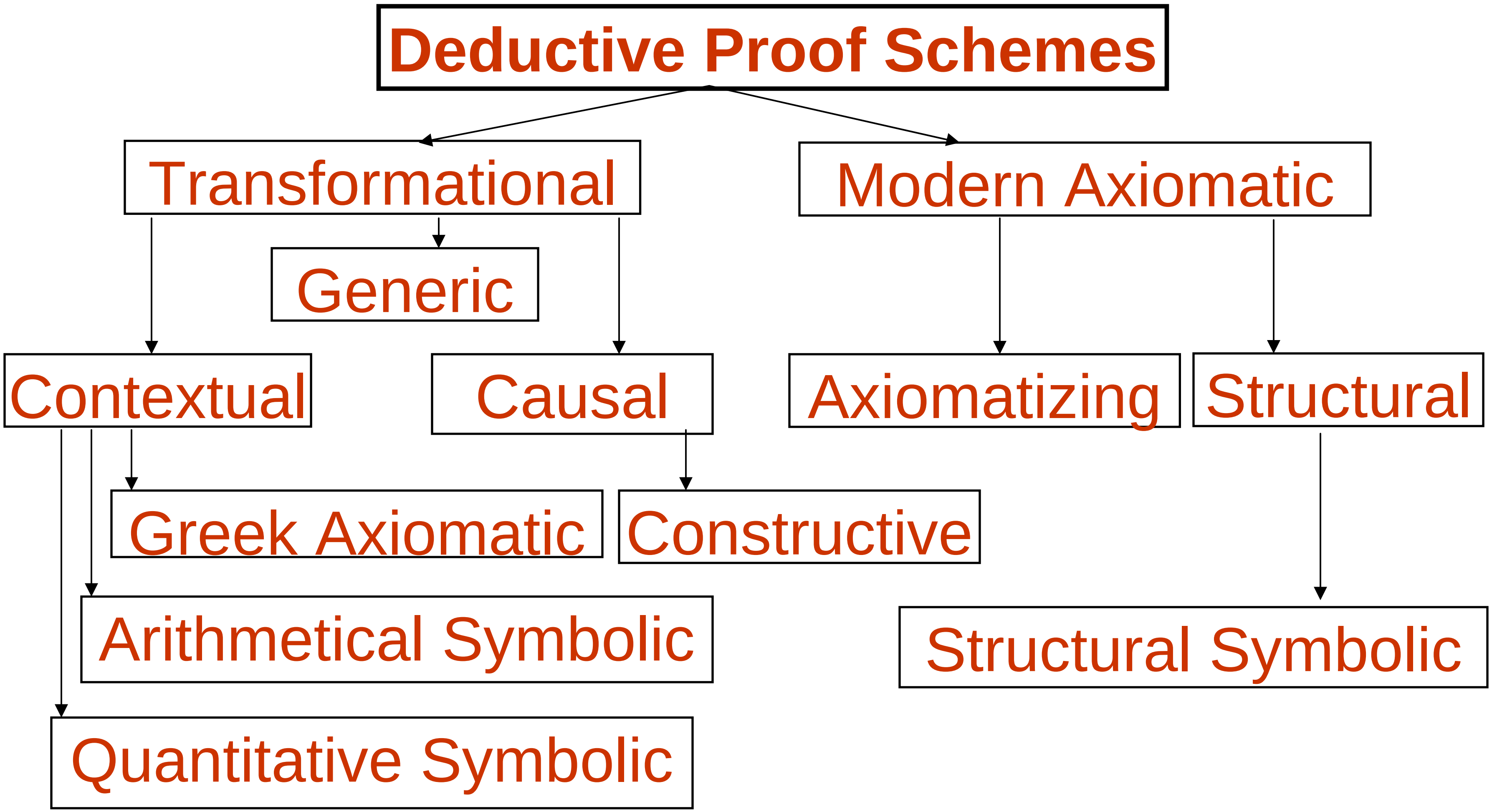
A *proof* is a particular justification one produces to ascertain for oneself or persuade others.

Proof Scheme

A person's ***proof scheme*** is a persistent characteristic of one's act of proving.

Taxonomy of Proof Schemes

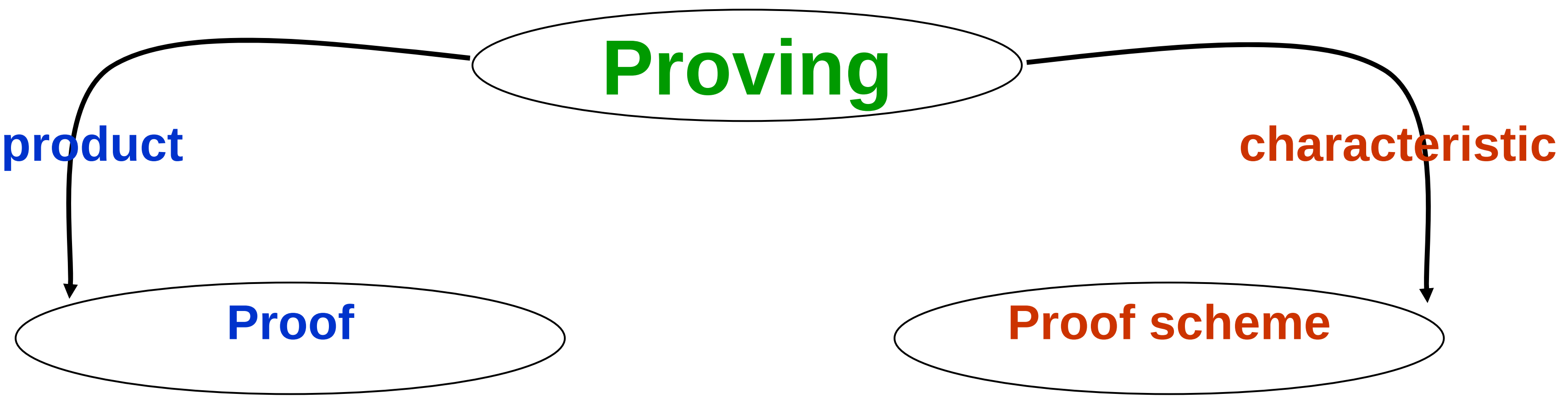



Deductive Proof Schemes
Transformational
Modern Axiomatic
Generic
Contextual
Causal
Axiomatizing
Structural
Greek Axiomatic
Constructive
Arithmetical Symbolic
Structural Symbolic
Quantitative Symbolic

Proving, Proof, Proof Scheme

A ***proof*** is a particular justification one produces to ascertain for oneself or persuade others.

A person's ***proof scheme*** is a persistent cognitive characteristic of one's act of proving.



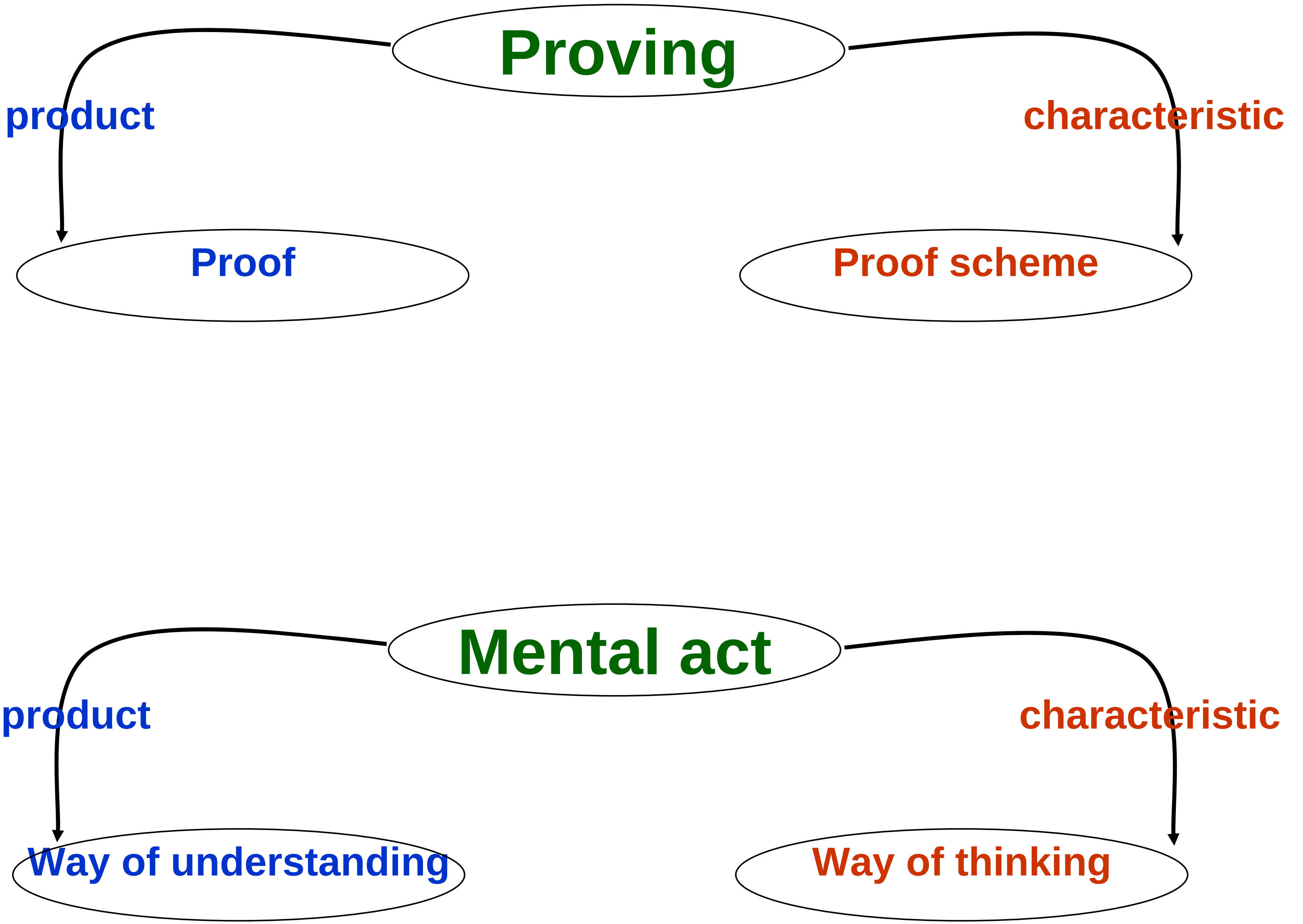
Mental Act

Humans' construction of knowledge involves numerous mental acts

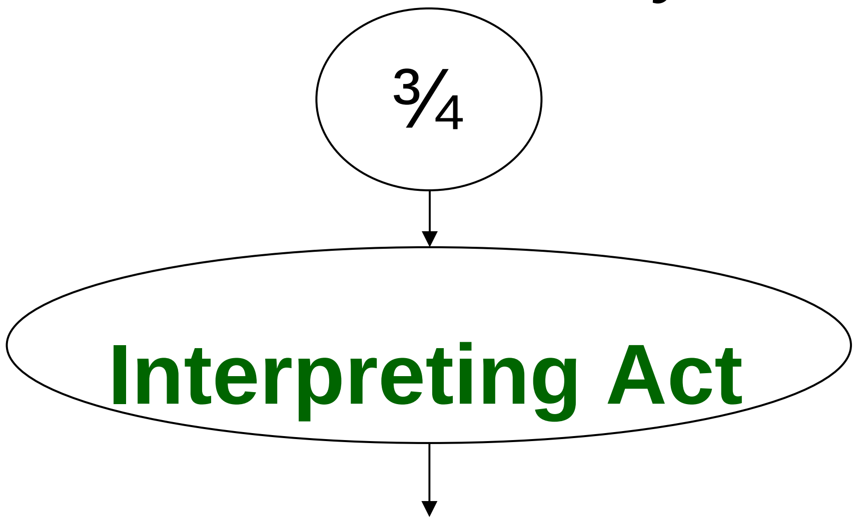
such as:

representing, interpreting, defining, computing, conjecturing, inferring, proving, structuring, symbolizing, transforming, generalizing, applying, modeling, connecting, predicting, reifying, classifying, formulating, searching, anticipating, problem solving.

Way of understanding **and** ***way of thinking*****: Generalizations of** ***proof*** **and** ***proof scheme*****, respectively**


Proving
product
characteristic
Proof
Proof scheme
Mental act
product
characteristic
Way of understanding
Way of thinking

A *way of understanding* is a particular *product* of *a mental act carried out by an individual*.



Ways of Understanding (products)

- Three objects out of four objects
- The sum $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$
- The measure of the quantity resulting from dividing *3* units into *4* equal parts
- The measure of a *3* cm long segment with a ruler whose unit is *4* cm long
- The solution to the equation $4x=3$
- The equivalence class $\{3n/4n \mid n \neq 0\}$
- Two numbers with a bar between them.

A way of thinking is *a characteristic of a mental act*.

Interpreting act

↓

Ways of Thinking (characteristics)

One's interpretation of symbols might be characteristically

1. inflexible: a symbol has a single interpretation
2. flexible: symbols can have multiple interpretations
3. non-referential: devoid of referents (quantitative, spatial, functional, etc.)
4. referential: a representation of an entity within a coherent reality

Mathematics **consists of two complementary subsets:**

- **The first subset is a collection, or structure, of structures consisting of particular axioms, definitions, theorems, proofs, problems, and solutions. This subset consists of all the institutionalized ways of understanding in mathematics throughout history. It is denoted by WoU.**
- **The second subset consists of all the ways of thinking, which characterize the mental acts whose products comprise the first set. It is denoted by WoT.**

$$M = WoU \cup WoT$$