Mathematical Habits of Mind and the language-learning brain: algebra as a second language

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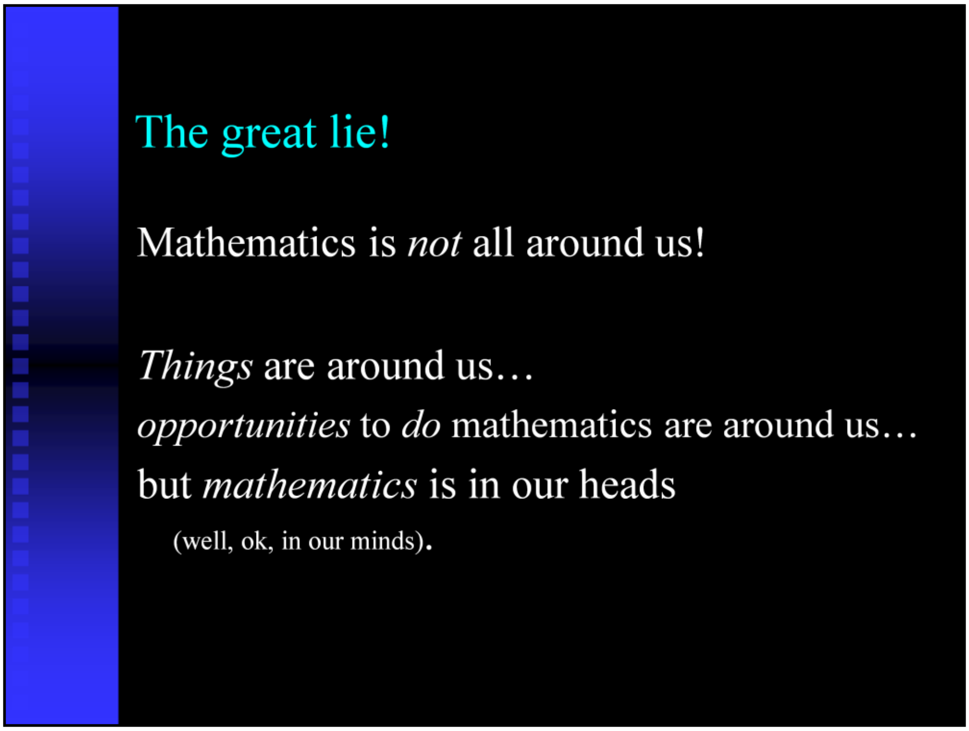
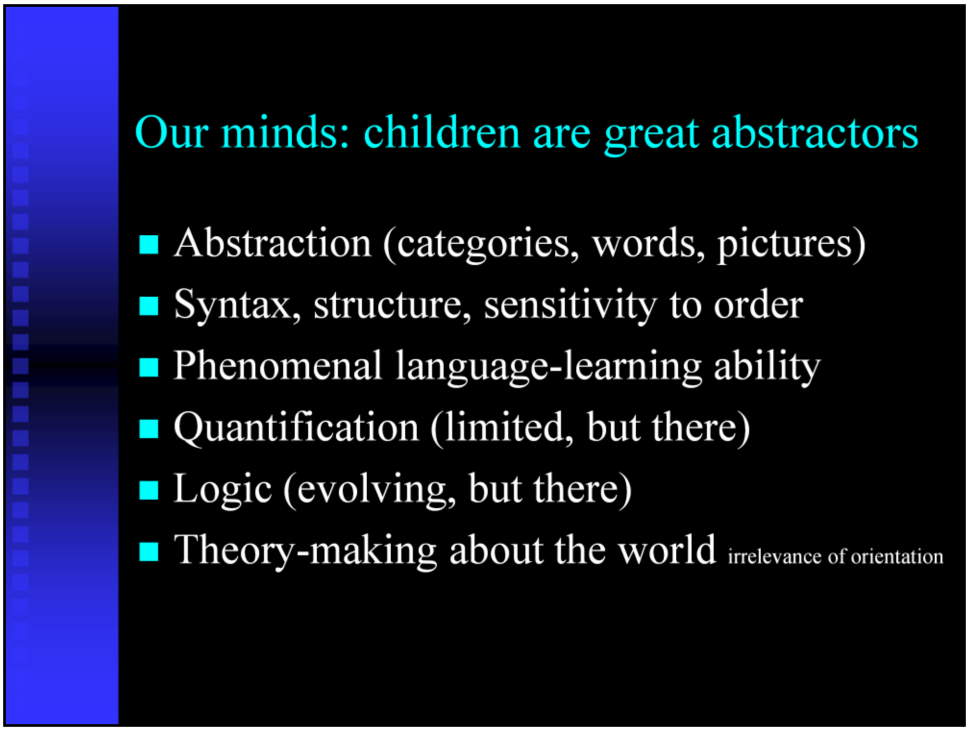
Fonts may align differently on different machines.

More details on http://thinkmath.edc.org

A paper providing all the filler details, background, and cited sources is currently in preparation.



And so, in order to talk seriously about mathematics and how it is learned by young minds— and, especially, about "mathematical habits of mind" — we must talk at least a bit about the nature of those minds.



ALL learning involves the building and adjusting of categories one abstracts from experiences around one. This is most easily noticed with children before they get to school, who experience a veritable tsunami of chaotic sensory input, filter and classify it in various ways, attach arbitrary sounds (words) to bits and pieces of it, and make all kinds of observations and claims that only they would ever have thought of. This is an act of abstraction and knowledge building not anywhere nearly as orderly as what we do in mathematics, but represents the mental ability without which mathematics would be impossible. Mathematics is a refinement of, not a departure from, this early abstraction which is absolutely natural and universally built in as our evolutionary version of teeth and claws.

Last bullet: orientation of objects is irrelevant (regardless of spatial orientation, the object is recognized as the same) in *all* of life. Except print! In print, b, p, q, and d are *not* the same object, despite being the same shape rotated or reflected, and WAS ≠ SAW. Mathematical *concepts* (at least through early high school) mostly behave according to the physical world, with order and orientation playing only the roles they do with objects. Mathematical *notation* (a syntactic world) is where one must think twice about order. Commutative property is *not* about shoving objects together in either order on a table: it is a property of abstract tokens in a logical system. The “semantic sense” of addition having this property can be illustrated by moving objects, and that is *all* that should probably be done with children, but the value is only to help

Algebra: ideas, language, tool

Some algebraic ideas *precede* arithmetic

Developmental

- w/o rearrangeability 3 + 5 = 8 can't make sense
- *Nourishment* to *extend/apply/refine* built-ins
  - breaking numbers and rearranging parts (any-order-any-grouping, commutativity/associativity),
  - breaking arrays; describing whole & parts (linearity, distributive property)
- But many of the basic intuitions are built in, developmental, not "learned" in math class.

Before "conservation of number" begins to kick in (roughly between ages 4 and 6), quantity isn't always stable for young children. At this stage, even if a child can successfully count eight objects, the child may tenaciously insist that when those objects are spread out, they are "more" than when they're bunched closely together (and, if they are candy, will want the "more" even if, upon actual counting the two groups, the child counts higher on the "smaller" group). If 8 doesn't always equal 8, then the assertion that 5 + 3 = 8 has no logic to it at all. Neither do our names have logic, of course, and we're capable of remembering them, but we don't *want* mathematics to be built as a collection of arbitrary and illogical statements.

The pre-conserving child may "know" that 5 + 3 is "more" than 8, for example, and it doesn't serve a learner's sense of depending on reason to say, in effect, "your reasoning is inadequate, so listen to mine." So, in order to *learn* arithmetic, the child must *first* acquire the algebraic ideas behind pulling apart and rearranging and reassembling the parts of numbers — essentially an amalgam of what we call commutativity and associativity, but in the physical world of objects which doesn't need those properties to be differentiated (or even articulated). Algebraic *ideas* – at least some of them – *precede* the ability to learn arithmetic. They develop "naturally" with the child and not from instruction or cultural heritage; they're how we are evolved to make sense of the world.

Algebraic language, like any language, is *Convention*

- Children are phenomenal language-learners
- Build it from language spoken around them
- Infer meaning and structure from use: not explicit definitions and lessons, but from language used in context
- Where “math is spoken at home” (not drill, lessons, but conversation that makes salient logical puzzle, quantity, etc.) kids learn it

By contrast, language is *always* convention. We’re built to be adept at learning it, but it *is* cultural and requires experience. Still not “instruction” but use in context; it does not arise spontaneously from experience only with the world; it requires experience with native users of the language.

Language and computational tool

To *us*, expressions like $(n - d)(n + d)$ can be *manipulated*

- to *derive* things we don't yet know, or
- to *prove* things that we conjectured from experiment.

We can also use such notation *as language* (*not* manipulated)

- to *describe* a process or computation or pattern, or
- to *express* what we already know (or conjecture), e.g. $(n - d)(n + d) = n^2 - d^2$

Claim: While most elementary school children cannot use algebraic notation the first two ways, as a computational tool, most *can* use it the last two ways, as language.

Mathematical language; mathematical thinking

- Linguistics and mathematics
- Algebra as abbreviated speech (Algebra as a Second Language)
  - A number trick
  - "Pattern indicators"
  - Difference of squares
- Reading two dimensional information
- Word problems
- Developing mathematical language

Linguistics and mathematics

Michelle's strategy for 24 – 7:

(breaking it up)

Algebraic ideas

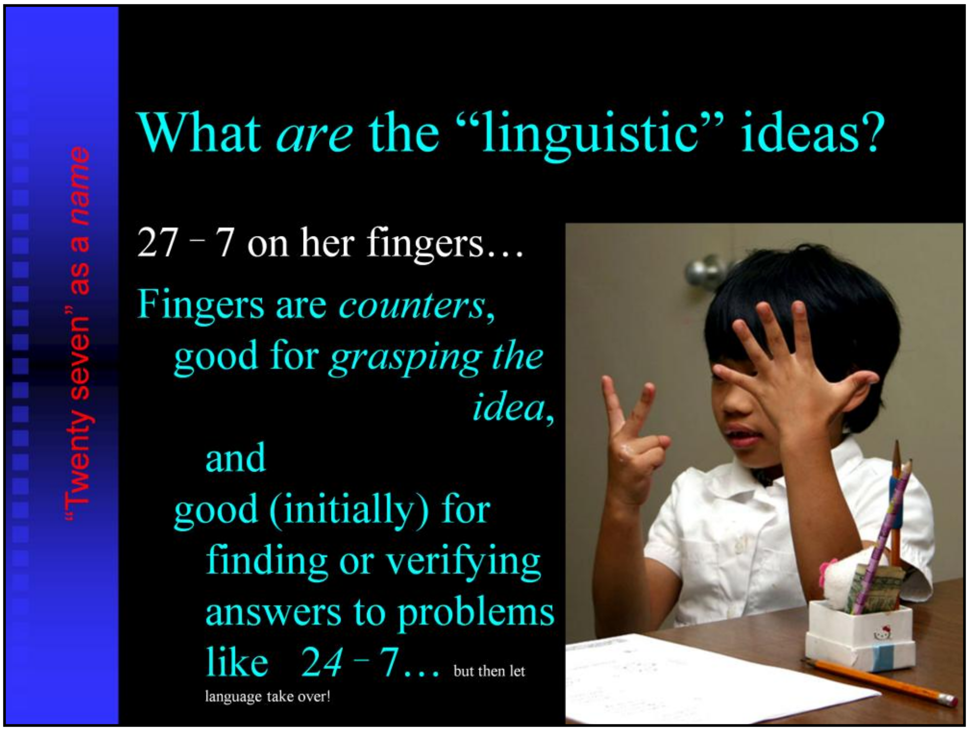
A *linguistic* idea (mostly)

- Well, 24 – *4* is easy!
- Now, 20 minus *another* 3…
- Well, I know *10* – 3 is 7,

  and 20 is 10 + 10,

  so, *20* – 3 is 17.
- So, 24 – 7 = 17.

Arithmetic knowledge



Any countable objects — and fingers are cheap, portable, always available — are good for beginning to understand the *idea* of addition and subtraction, which arise first as shortcuts for counting forward or backward. When children are just getting started with arithmetic, fingers (or other counters) are also useful for finding or verifying answers to certain problems, like 27 – 4.

But the way we’ve named numbers is intended to make certain problems, like 2***4*** – 4, feel as automatic and effort-free as language is. We’ve named numbers — they weren’t born with names! — to say what they’re made of. German names 24 “four and twenty”; English names it “twenty-four”; but either way, we see it has two parts. We *could* count backwards to take 4 from 24, but we could also just use the name: taking “four” from “twenty-four” is like taking “Lennon” from “John Lennon”; we don’t have to “do” anything to know what’s left.

For more about why the 24-4 part is a *LINGUISTIC* notion,

see http://thinkmath.edc.org/index.php/Addition_and_subtraction#Adding_and_subtracting_10

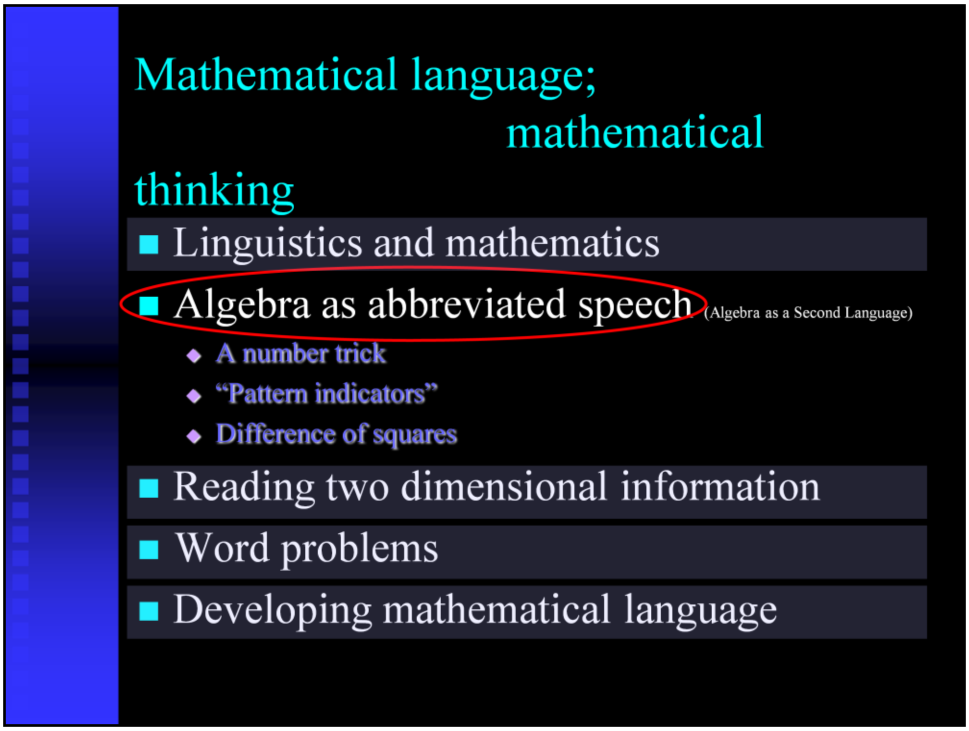
“Five hundred” and “five sevenths” as *names*

Five hundred plus two hundred

- Our *language* makes this “obvious” to children.
- And ask first graders about five ninths plus two ninths.

The *symbols* may confuse children, but the *language* does not, and so if we teach the ideas *linguistically* first, and then use the symbols as a way of *writing what the children already know*, their learning will be much easier.

The idea of reading and writing across the curriculum can be good, but one wants to temper it with a recognition of how difficult it really is, and not let it *compete* with the other learning that the child is doing.



Elementary algebra serves two important purposes: expressing numerical computation, patterns, and relationships that we already know; and deriving various things that we do *not* already know. These are not important to distinguish when we are teaching algebra to older students and adults: both purposes feel “central” to the subject, and both uses are possible for adults to learn. For children it is different. They are not, in general, able to use purely syntactic (formal) operations on symbols to derive things that they do not know. But they are the consummate language learners, and are *very* capable of using *this* language to express things that they do already know. Teaching “algebra” the way we might teach it to a 9^{th} grader is not appropriate, and not likely to work, for a 5^{th} grader. But teaching algebra as a second language—or, rather, as an abbreviated way of writing what the child might otherwise say—makes perfect sense. It makes life *easier* for them immediately, as an easier way of expressing what they are already seeing and trying to talk about, and it gives them an enormous advantage when they learn the other use of algebra a few years later.

Algebra as abbreviated speech (Algebra as a second Language)

- A number trick
- "Pattern indicators"
- Difference of squares

Surprise! You speak algebra!

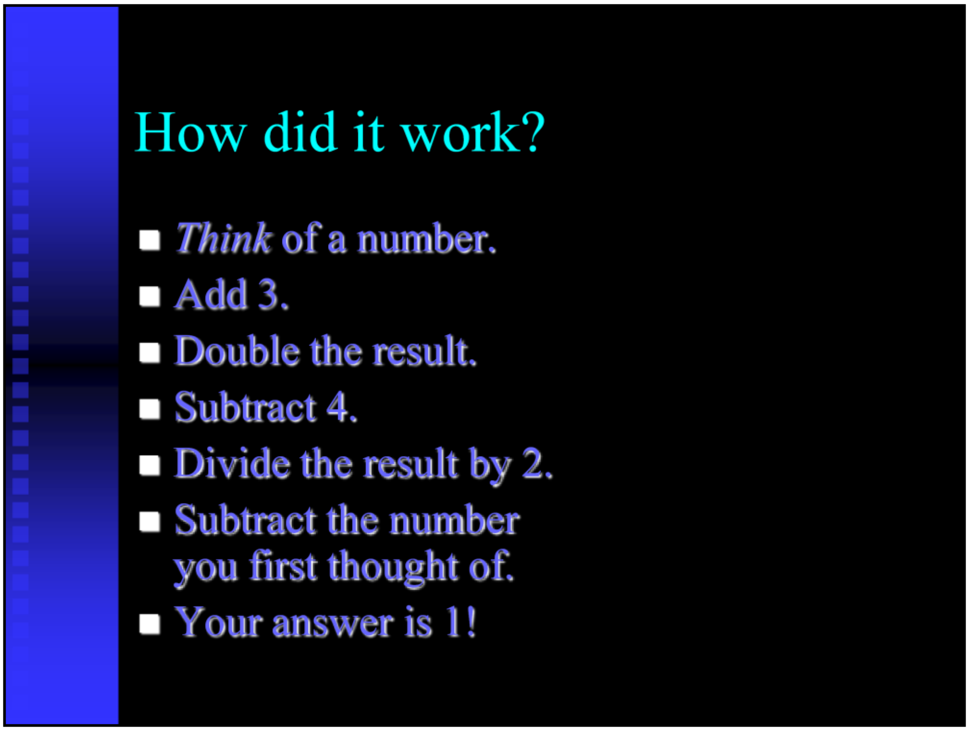
4th grade

A number trick

- *Think* of a number.
- Add 3.
- Double the result.
- Subtract 4.
- Divide the result by 2.
- Subtract the number you first thought of.
- Your answer is 1!

Kids LOVE doing these. First time 4th graders see this, they want to *memorize* every word, do it on their friends…

For more, see: http://thinkmath.edc.org/index.php/Number_tricks



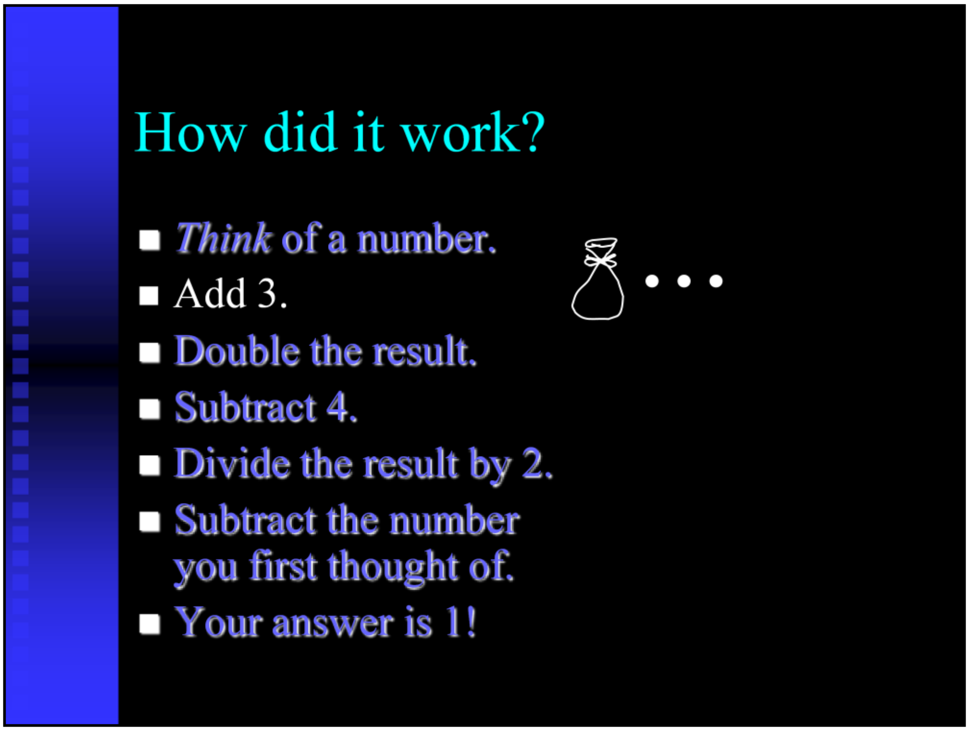
They also want to understand!

Before the development of algebra, people did try to explain how mathematical processes like this worked, using spoken and written language, and the results were cumbersome and very hard to follow. It is hard to ***explain*** how this trick works, but it is easy to ***show***.

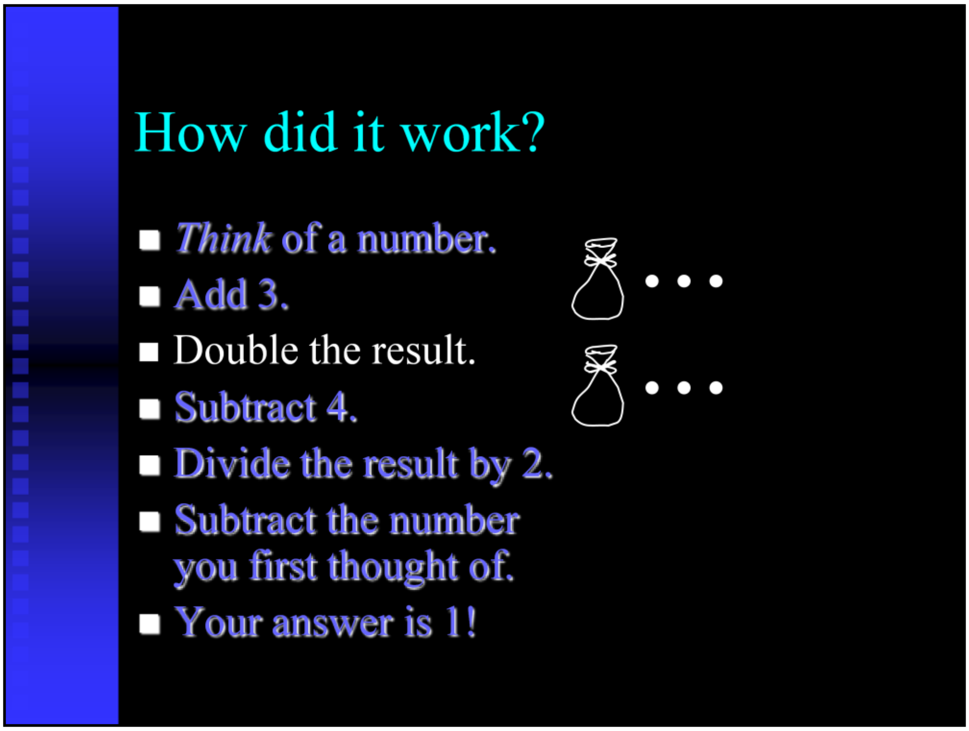
How did it work?

- *Think* of a number.
- Add 3.
- Double the result.
- Subtract 4.
- Divide the result by 2.
- Subtract the number you first thought of.
- Your answer is 1!

I imagine a bag with that many marbles in it, tied with a string, so that only you know how many are in there.



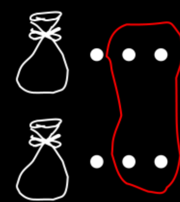
I still don’t know your number, but I can draw a picture that represents adding 3 to your number.



And I know how to double. I still don’t know your original number, but I know this represents the number you *now* have.

How did it work?

- *Think* of a number.
- Add 3.
- Double the result.
- Subtract 4.
- Divide the result by 2.
- Subtract the number you first thought of.
- Your answer is 1!



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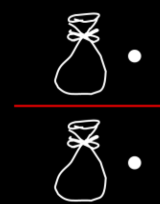

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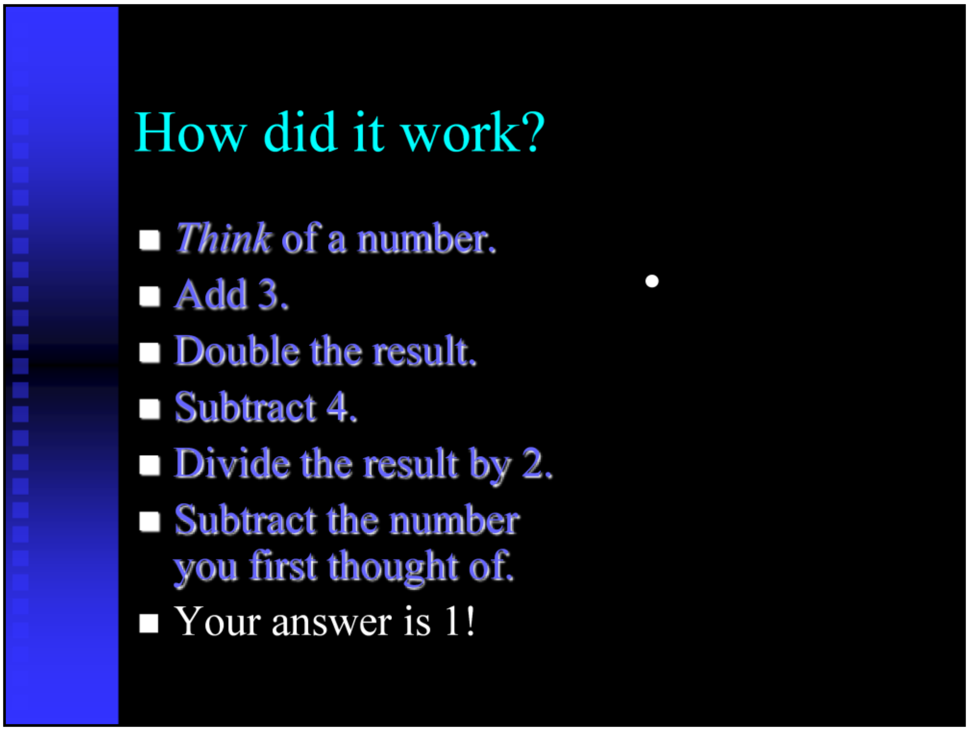


The instructions are a recipe for (2(n+3) – 4) ÷ 2 - n.

The result, 1, is an algebraic simplification of those instructions.

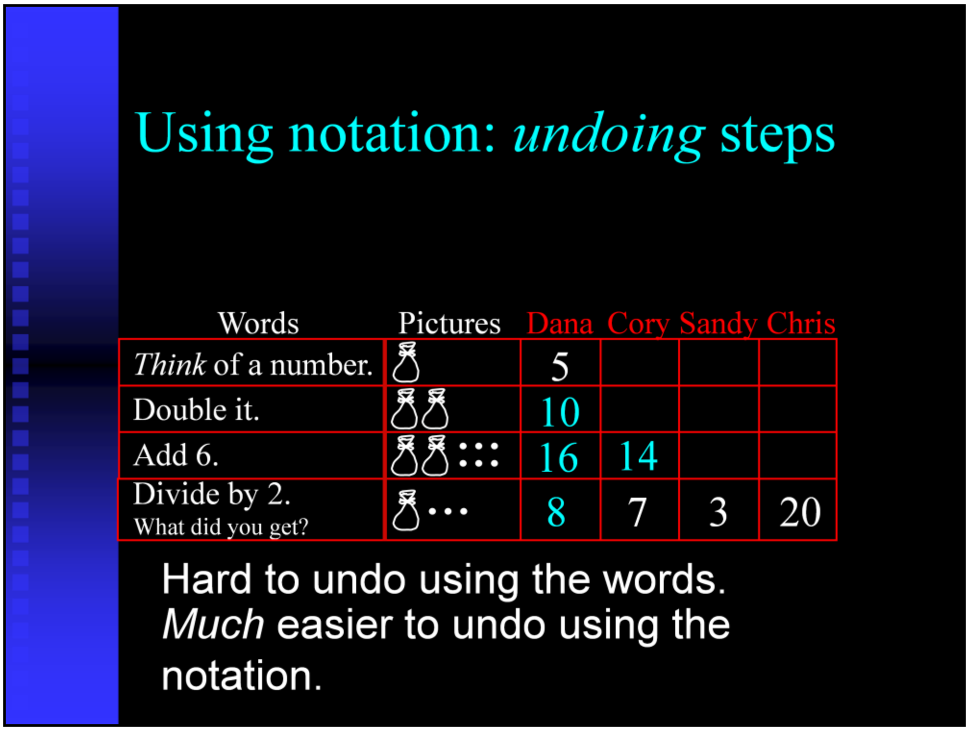
Kids need to do it themselves…

Using notation: *following* steps

Words	Pictures	Dana	Cory	Sandy	Chris
Think of a number.		5			
Double it.		10			
Add 6.		16			
Divide by 2. What did you get?		8	7	3	20

They get some practice with arithmetic, but this practice is incidental to what else they're doing, which involves mathematical language, using tabular format and, most important, learning (via this "trick") both the kind of structure that one finds inside a multi-step arithmetic process, and the kind of symbolic system that helps record that structure in an efficient and clear way.

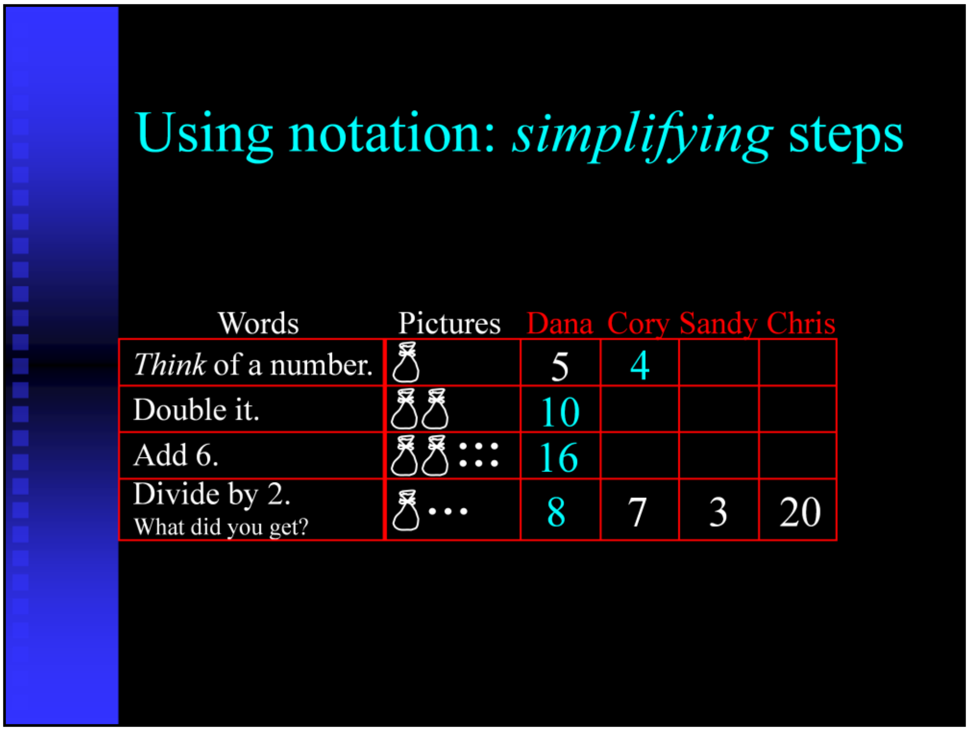
Incidentally, this also shows the two-dimensionality of mathematical "reading." The significance or nature or meaning of any entry in any cell depends on both column and row.



With Dana, we knew what the starting number was, and already know how to follow instructions to get from there to the end. With the other children, we know how they *end*, and need to work backwards.

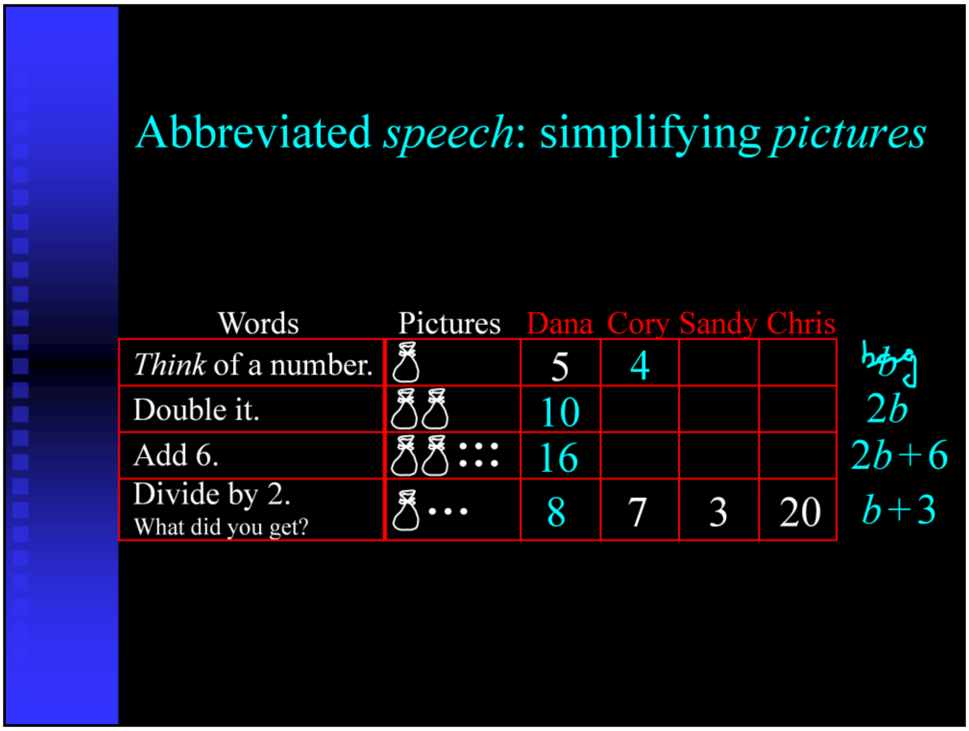
If we try to do that with words, alone, it is hard! To *do* the steps, we read the instruction *on the same line in which we write the answer.* It is natural that to *undo* the steps, even adults are easily mixed up. To fill in the number just above the 3 in Sandy's column, we must read the instruction on the line with the 3, not the line we intend to fill in, and we must not *follow* the instruction, but perform the opposite.

But if we do it with the picture notation, the job is much easier. We can, again, read the "instructions" (pictures) on the line we wish to fill in, and compare them with the pictures on the line we already know. Then, we can either work backwards step by step, or…



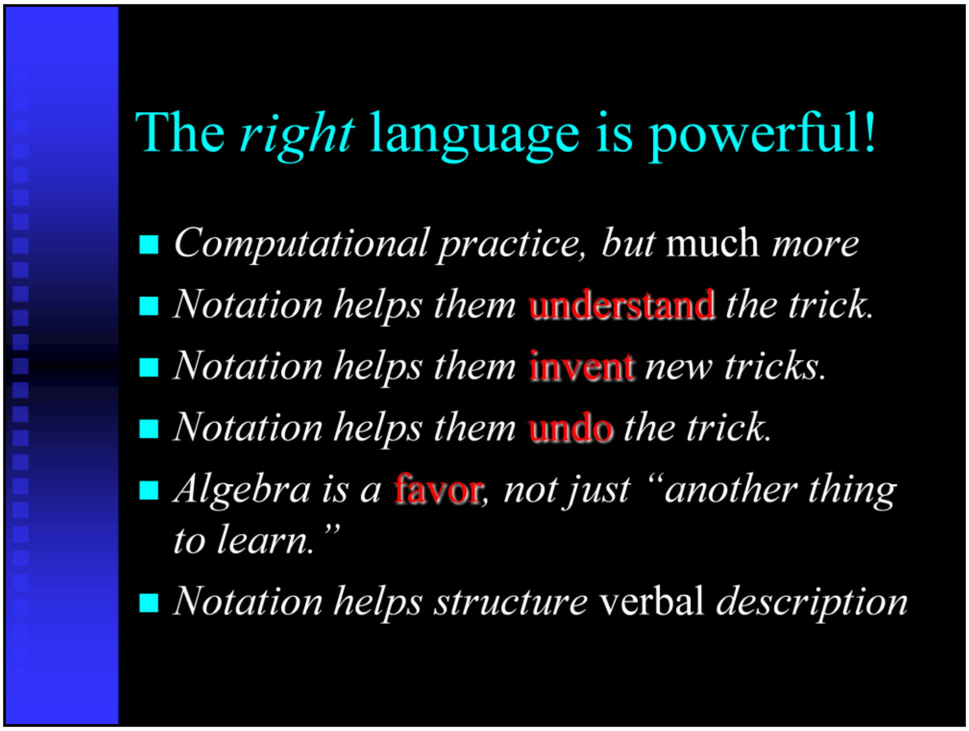
…we can “jump” directly to the beginning. If Cory’s 7 is one bag and three marbles, then the bag, itself, must contain 4 marbles.

Later, as we keep drawing marble bags on the board, they become a nuisance to draw, so we simplify the picture. No more tie around the opening of the bag. No more bag top. No more bag bottom! Only the x is left where the tie used to be.



Scrawl “bag” first, to make clear that we are describing the pictures. But it’s neater and easier just to abbreviate to “b.”

NOT about “letters standing for numbers” or about “variables” or about “algebra.” This is just a way of abbreviating ENGLISH. A picture is worth a thousand words. A word is easier to write. An abbreviation is even easier!



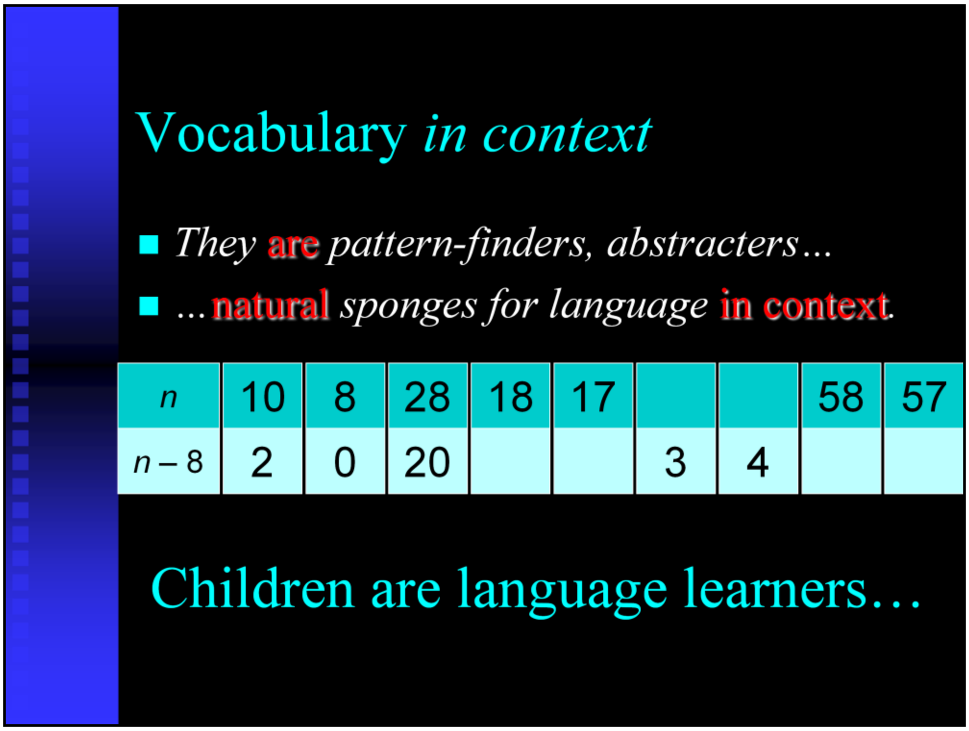
The most important message is that there *can* be a more powerful notation than words, and that some mathematical acts become easier if we find a suitable notation.

Too often, algebra is just "another thing to learn," not at all a favor to kids. At this early age, algebra should be a convenient way to record what they already know, and to help see the results of processes. Syntactic manipulations of algebra, at this age, are rarely appropriate -- kids can't use algebra to derive or prove what they don't already know -- but they *can* use the symbols of algebra to record what they *do* already know. It is a language, and kids are great language learners, when the language is used sensibly in context!

There is an extra bonus. We do want children to be able to describe their mathematical thinking verbally and in writing. Giving them the *appropriate* language helps them do that. And useful notation can even help structure their *thinking* well enough to allow them to describe it more easily. We have to understand *before* we can describe. Attempting to do these two hard tasks at the same time sacrifices both of them.

Algebra as abbreviated speech (Algebra as a second Language)

- A number trick
- "Pattern indicators"
- Difference of squares



Michelle, a second grader, was typically fast in figuring out puzzles like this. Very early in the year, she saw this table, figured that it was about subtracting 8 (top to bottom) or adding 8 (bottom to top) and had already finished the puzzle before I had the chance to hand all the papers out. “How did you do it so fast,” I asked. “I figured it out, but I didn’t even have to. It says it right there.” She smiled as if she’d discovered a secret-code answer key that we had accidentally left on the paper!

But she wouldn’t have known that “it says it,” if she had not *already* figured out the pattern. Only *after* she knew the pattern would the symbols $n - 8$ make any sense to her. No attention at all was ever called to the algebraic symbols in class, no teaching of them, no mention of variables, no mention of letters instead of numbers, or “this means *any* number we want.” No mention at all. Just by looking at the numbers, Michelle derived the pattern of the page: “subtract 8.” She then attached that meaning to the “pattern indicator” at the left without so much as a word from the teacher. She was not told the meaning of the algebraic symbols, but invested those symbols with that meaning, and then even “saw how” the “subtract 8” showed up in those symbols. That is how all children learn virtually *all* of their spoken language. A dog passes by, mommy makes the sound “doggy” and baby attaches *that* sound to *that* sight. Baby may generalize incorrectly at first (applying “doggy” to all small animals, for example), but eventually sorts it out.

Algebra as abbreviated speech (Algebra as a second Language)

- A number trick
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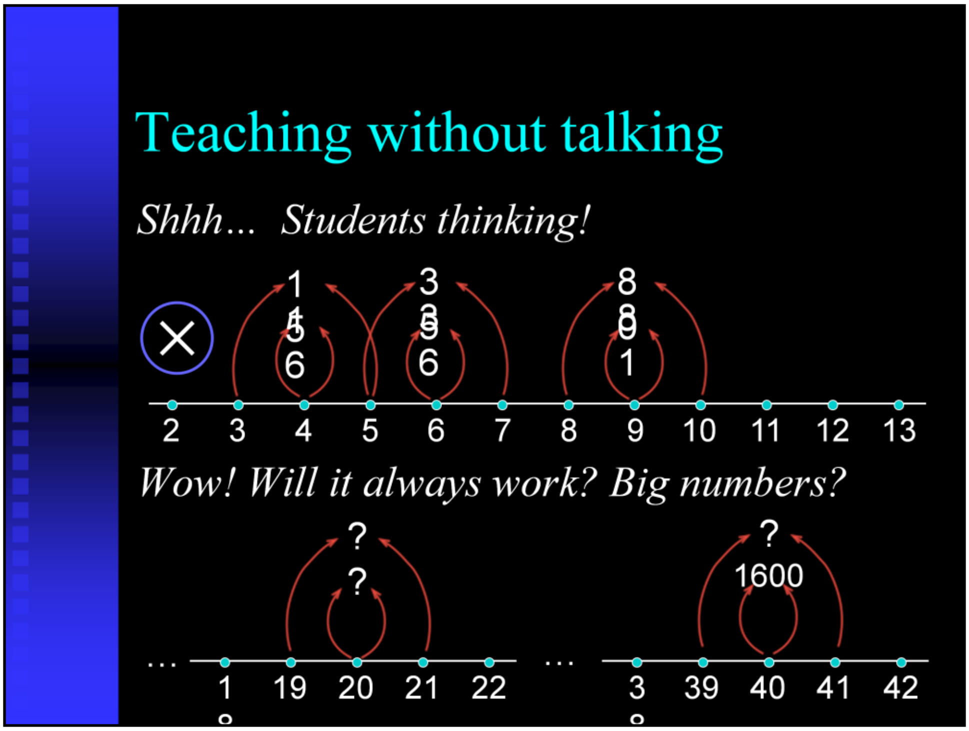
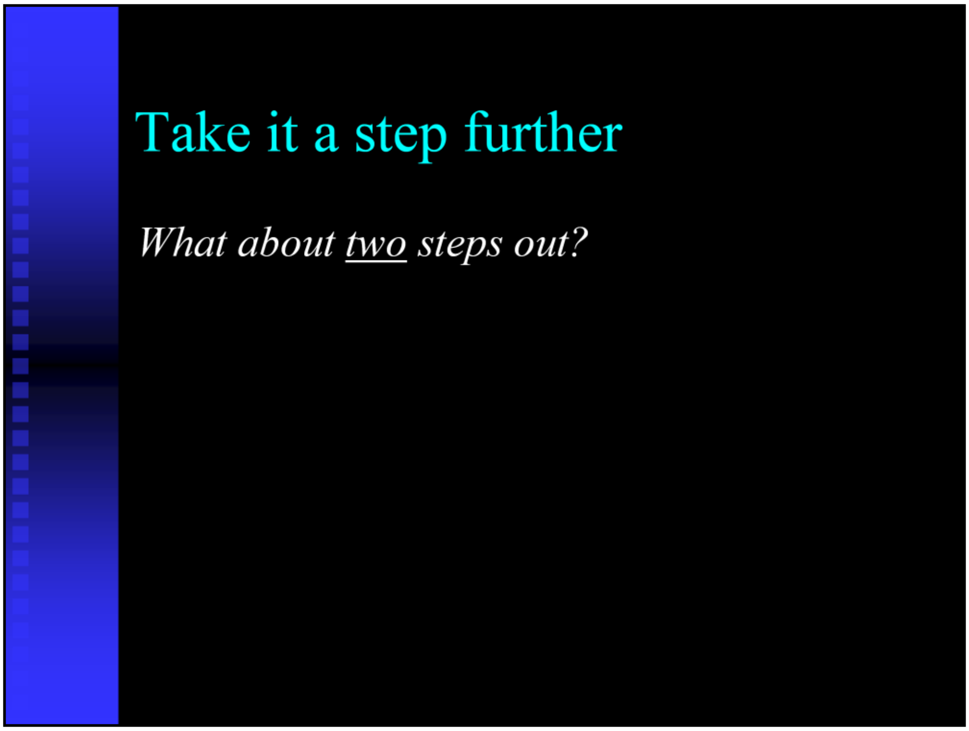


Multiplication “onions” or “rainbows.” This activity is conducted with the *teacher* not talking at all (it’s nice, but not necessary, to have children not talking either) and is suitable at *any* grade in which the children collectively know all the basic multiplication facts, even if none of the children are, individually, fluent with all of them.

The teacher puts up a number line, draws a pair of arrows from any single number (in this example, I chose 6), fills in the product (a square), draws a pair of arrows from the two nearest neighbors to the left and right (5 and 7), fills in the product, draws a pair of arrows from any new number, and then pauses, offering the chalk or marker to any volunteer, assuming that *some* child will notice that the numbers are products. If nobody takes the challenge, fill in the product, and draw the near-neighbor arrows and again offer the marker. Continue for several more examples, until children are hopping out of their seats having “noticed something.” Options: If children need practice on facts and/or multiplication, challenge “will it always work? Try some other cases. Try a few big numbers (like 27 x 27 and 26 x 28). The practice is interesting because it is not arbitrary work, but part of discovery/research.

Then, return to the silent presentation, posing examples like those on the bottom, where a multiple of 10 is squared (easy to do in one’s head) and then its two neighbors are “multiplied mentally” because kids know how to find the product of these near neighbors.

“Wow! How did you know work 19x21 in your head so fast?! What about 49x51?” (again, show it on a number line with the 50 between, so



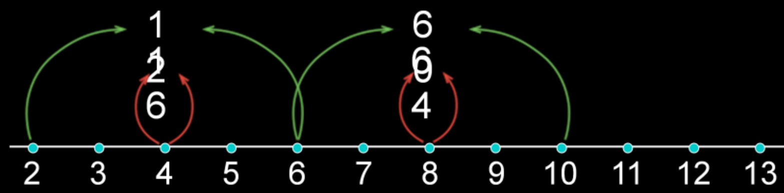
Then do the same thing "two steps away." E.g., 6x6 followed by 4x8. 8x8 followed by 6x10. 5x5 followed by 3x7.

If the class needs practice, give them time to find more cases. If they don't need practice, just go on to the neat challenge!

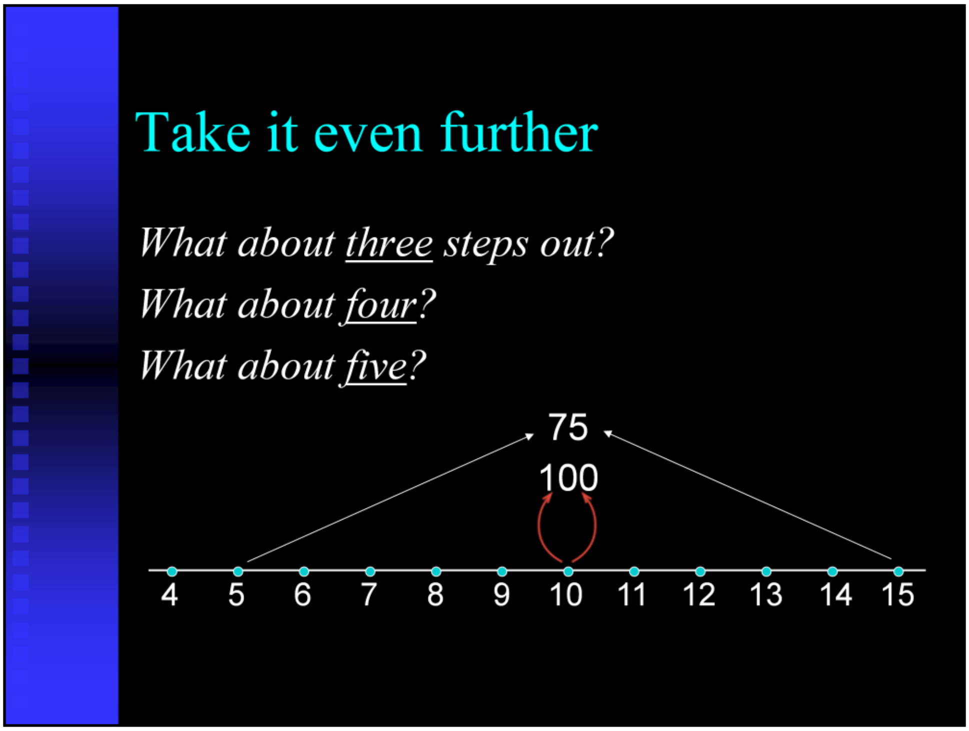
What's 28 x 32? If people don't see right away, draw in the arrows that mean "30x30." What about 45 x 62?

Teaching without talking

Shhh… Students thinking!



Again?! Always? Find some bigger examples.



For each new step, allow time to do the research, challenge kids to multiply some higher numbers, and look for a pattern.

What is the pattern in what we subtract?

1 step away: 1.

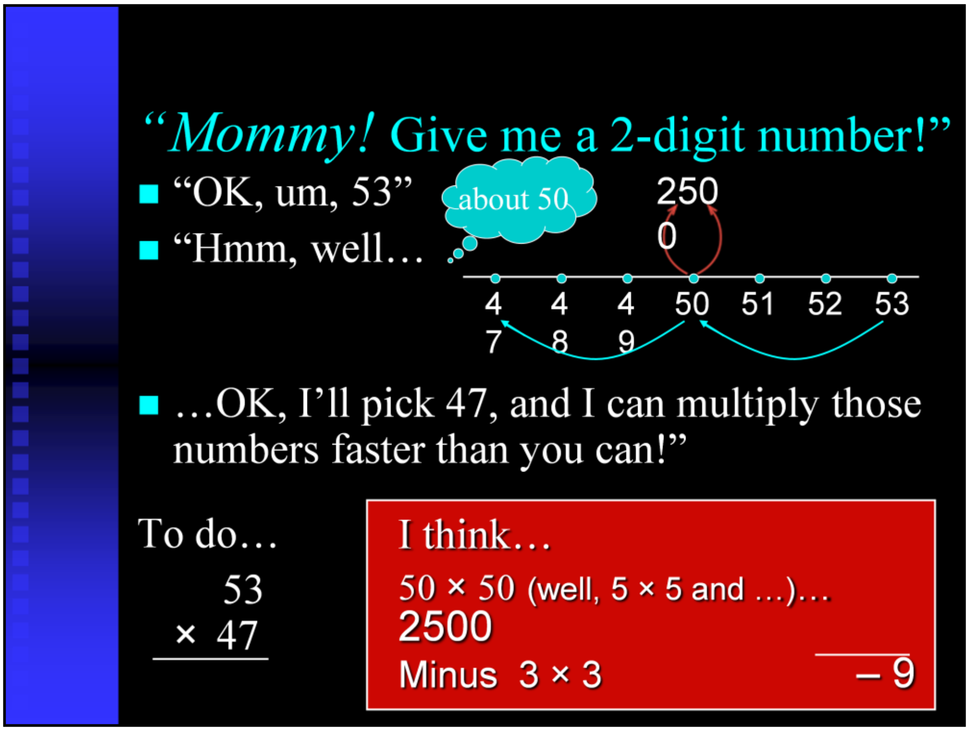
2 steps away: 4.

3 steps away: 9.

4 steps away: 16.

5 steps away: 25.

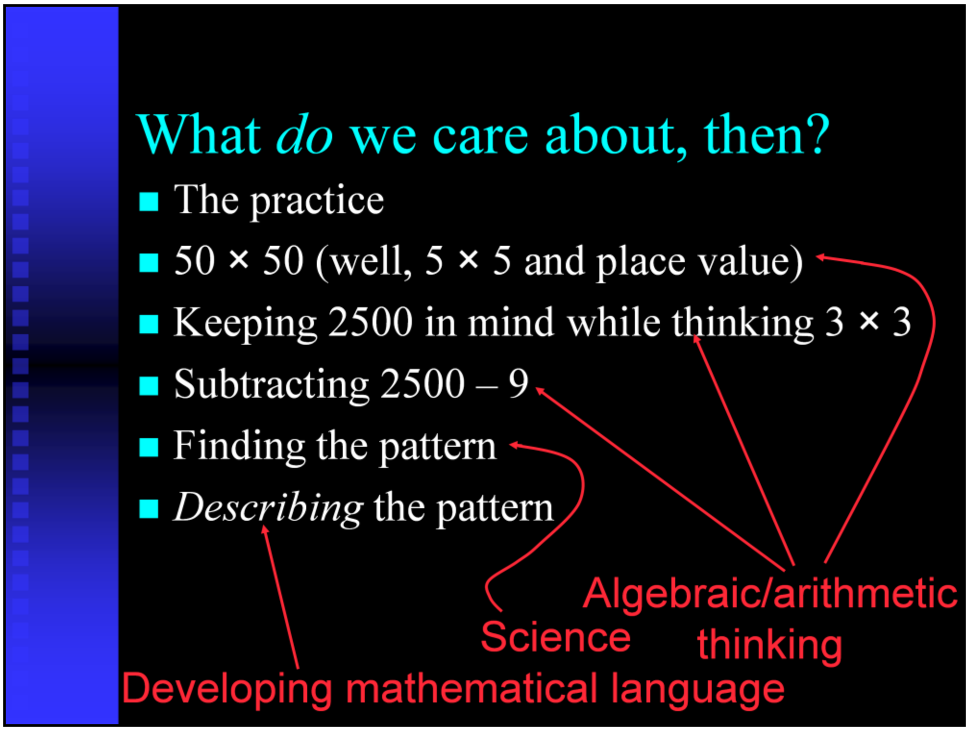
You can, if you want, then compare 30 x 40 to 35 x 35 (five steps away!). So we now have a way of finding the squares of 2-digit numbers ending in 5. Ah! And if you can do 35 x 35 in your head, then you can do 34 x 36, too, can't you!



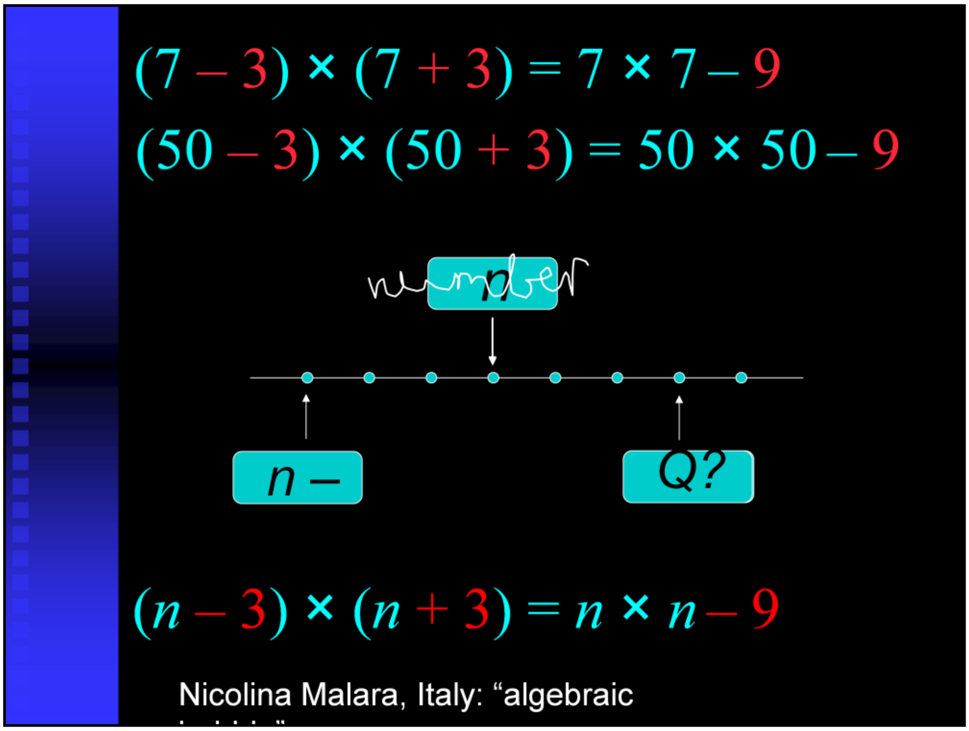
For the teacher, this is an opportunity to provide practice that feels spiffy and impressive to the child! Gives kids LOTS of practice, but always a novel feel, and the sense that they are being very smart. Kids show off to parents. "Give me a number under 100." Parent gives 84. Child then says, "OK. Multiply 84 x 76 and see if you can beat me." While parent is writing it down on paper, the child says "The answer is 6384."

This may also spark the teacher's interest!

But *nobody cares* if kids can multiply 47×53 mentally!



All the algebraic thinking also asks kids to develop bigger "mental buffers." See http://thinkmath.edc.org and search for "focus" or "attention."



To get from the algebraic “thinking” to the language, we (at elementary school) DON’T talk about “variables” and “letters standing for numbers” or “algebra” at all. We ask kids to express the pattern they’ve seen IN ENGLISH. It turns out to be VERY awkward to do. So we give the numbers names. See http://thinkmath.edc.org/index.php/Difference_of_squares for more about how this is done.

When teaching, I deliberately write fast (naturally producing my worst handwriting) to make the case that I’d like something simpler and with less writing involved!

Generalizing from a specific case (working various distances from 7) to a general case (any distance from any number).

The result -- not something we need to write in 5th grade, but something they *CAN* write then -- is a major topic in algebra! And they already know it, whether they’ve written it or not. They are “foreshadowing” an idea that often comes “hard.”

Nicolina Malara, a researcher in Italy, refers to this (and even more non-standard) algebraic notation as “Algebraic Babble” -- a kind of not-quite-grammatical but fully communicative precursor to the real thing, and as “safe” for children to pass through as natural language babble is as they

Make a table; use pattern indicator.

Distance away	What to subtract
1	1
2	4
3	9
4	16
5	25
d	$d \times d$

$$(n - d)(n + d) = n^2 - d^2$$

Again, the use of “d” is not a “variable” or an algebraic trick. It is simply an abbreviation for what we mean: distance. It is a way of making English EASIER, or rather, making it easier to say what we want.

Mathematical language; mathematical thinking

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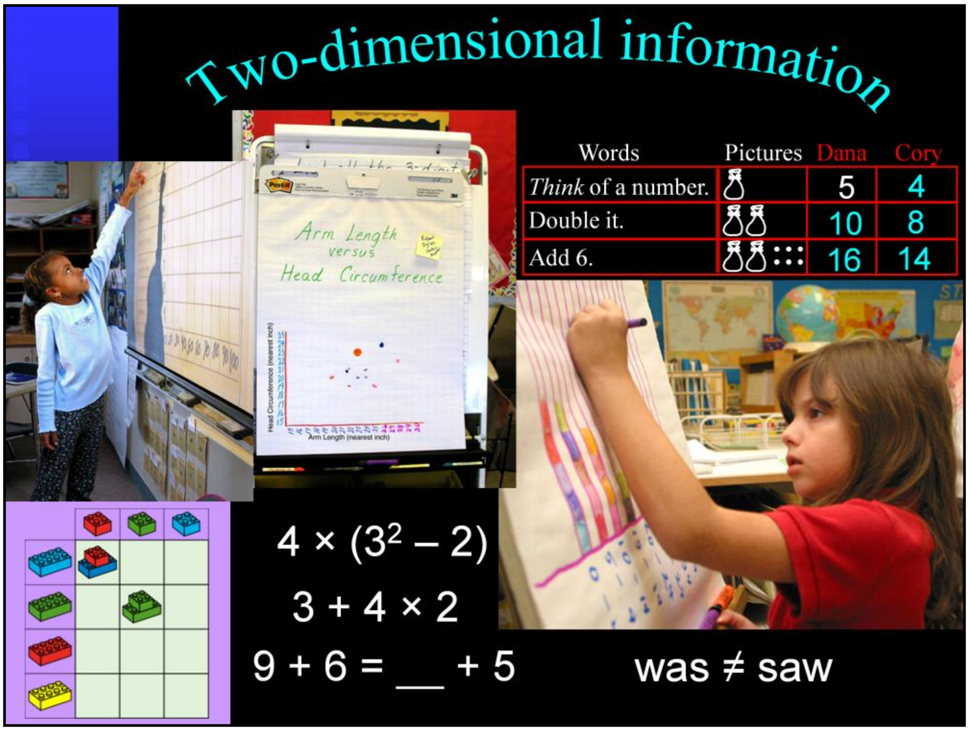
- Reading two dimensional information
- Word problems
- Developing mathematical language

Quick bail out!



Learning to read mathematics is different from learning to read English. Unlike English prose, which reads strictly left to right (the up-down component is merely a concession to page width), mathematical information is VERY often two-dimensional in nature. Coordinate graphs require two dimensions of information (the horizontal and vertical position of the dot); bar graphs coordinate two dimensions (which bar and how tall) in order to convey information; charts; tables….

Even symbolic notation uses vertical information (superscripts like the 2 in 3^2, or numerator and denominator in a fraction) as well as horizontal information.

Moreover, even the horizontal information is not strictly left to right. Students need to know that even before they learn order of operations (e.g., to do 9 + 6 = __ + 5).

When infants first learn to reach for a bottle, they don't recognize the bottle if it's presented upside down (in the wrong orientation), but they very quickly learn that it doesn't matter which way an object is held; it's the same object. That remains true pretty much throughout life! And for some children, that's very hard to unlearn, when they encounter PRINT. Reading presents an exception: d ≠ b ≠ p ≠ q. Orientation matters, and order matters, too: was ≠ saw. It takes time to learn that *print* is different from virtually everything else. Presented with 8 objects on a

Naming intersections, first grade

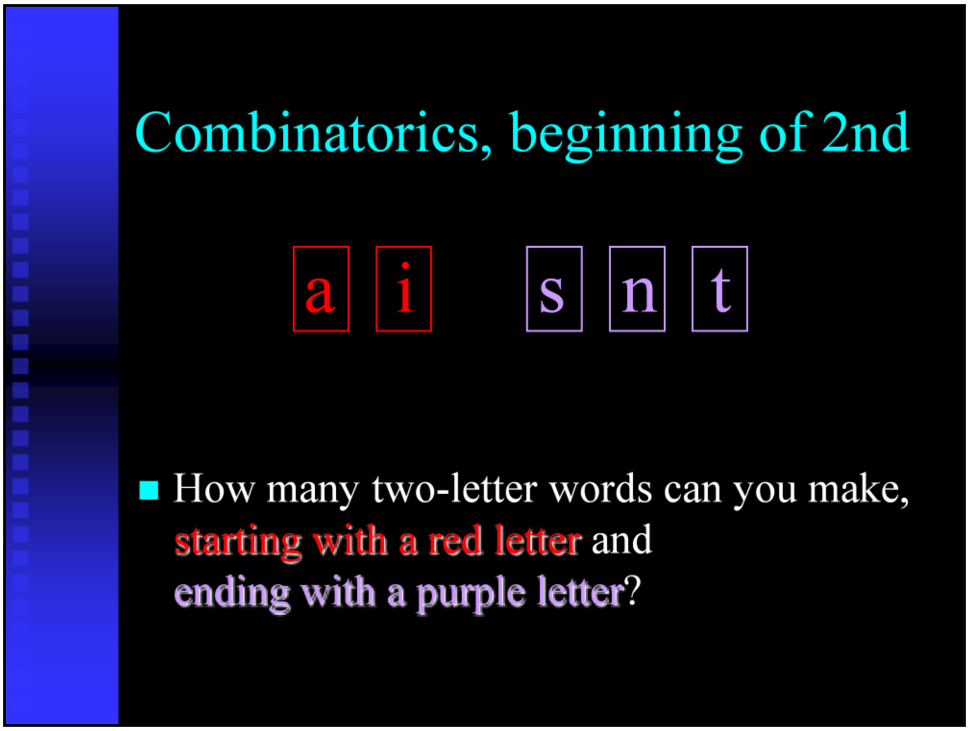
Put a red house at the intersection of A street and N avenue.

Where is the green house?

How do we go from the green house to the school?

So we start teaching about mathematical reading early. These first graders are getting an introduction to geometric/spatial ideas and the language for describing location. Because kids get words in context, they often don't even wonder what "intersection" means—they figure it out from context—even if they've never heard the word before. But if they're not sure, "at the corner" helps clarify, and then, totally naturally, we go back to using the fancier word.

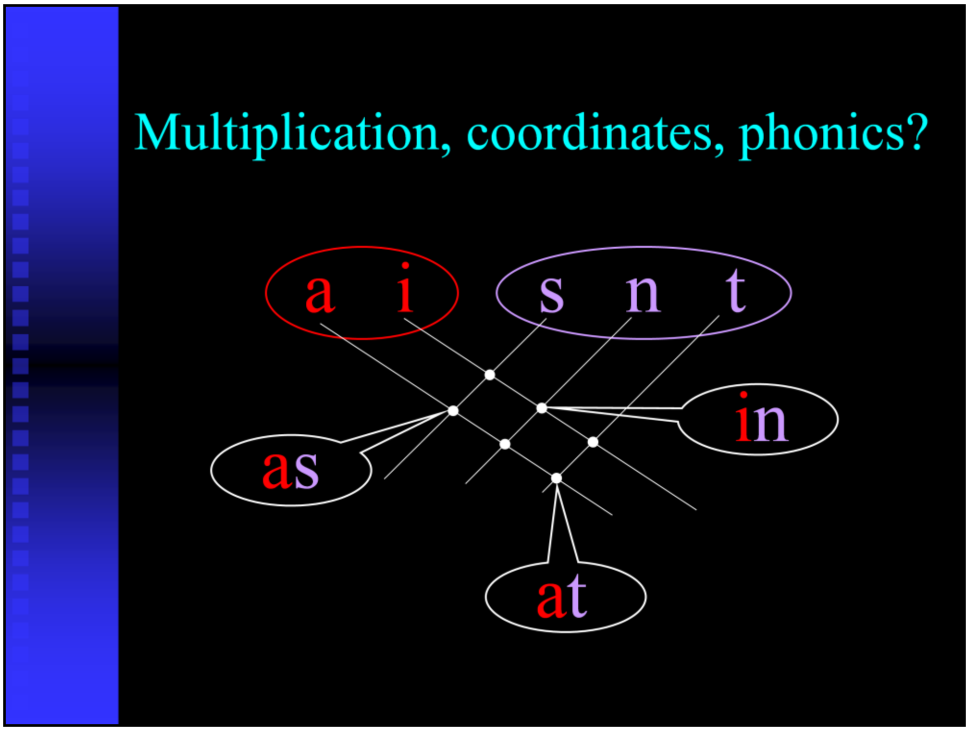
When we ask "where is the green house" kids often point and say "there!" To encourage the language, pretend you're asking over the phone and can't see what they're pointing to, a quite realistic scenario!



Among the earliest lessons in second grade, children are learning to be systematic in making lists. It takes a very long time. It is not mastered in a lesson, or a week of lessons (which is why we start early), but they can do it experimentally and erratically, but with fun and satisfaction right at the start, and gain competence slowly over time. (Compare learning to talk, or ride a bicycle.)

Children have actual cards with letters they can rearrange. They begin non-systematically but, because there are only a few combinations, they all succeed in finding all the combinations.

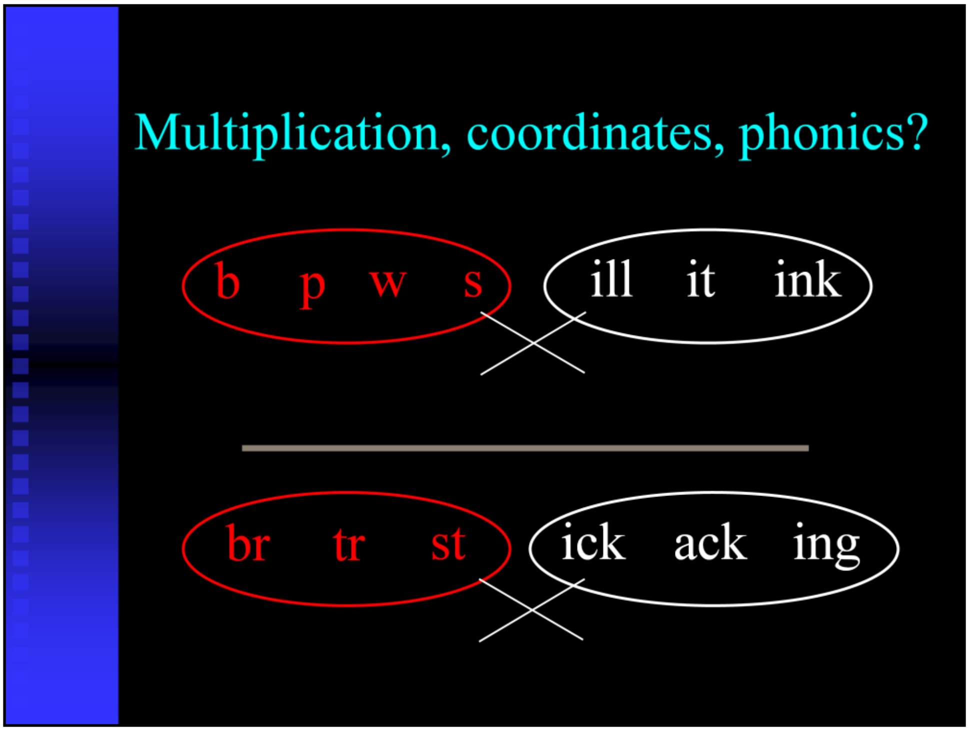
For the teacher, several unexpected connections between math and language arts: classification (vowels and consonants), combinations (multiplication on the math side, phonics on the LA side). Note that “how many letters” calls for addition. “How many two-letter words” (in a particular order) calls for multiplication.



After they have explored, we make it systematic with a metaphor of streets and avenues. “Show a car driving (or Drive your finger) along ‘A street.’ Now drive that car along ‘S avenue.’

There’s a stop light at every intersection in this tiny town! Can you show the intersection of ‘A street’ and ‘S avenue’? If kids are uncertain about the language (a rare occurrence), rephrase: Show where ‘A street’ and ‘S avenue’ meet, the intersection of ‘A street’ and ‘S avenue’.” Each stoplight is named by the intersection it is at. This stoplight is named IT. Can you find the stoplight named AS? What is the name of *this* stoplight? (AT)… etc.

When we ask how many roads (letters) altogether, that’s addition. When we ask how many intersections (letter combinations), that’s multiplication. Is this practicing phonics, or coordinates, or anticipating multiplication? Or systematic listing, or combinatorics? Yes.

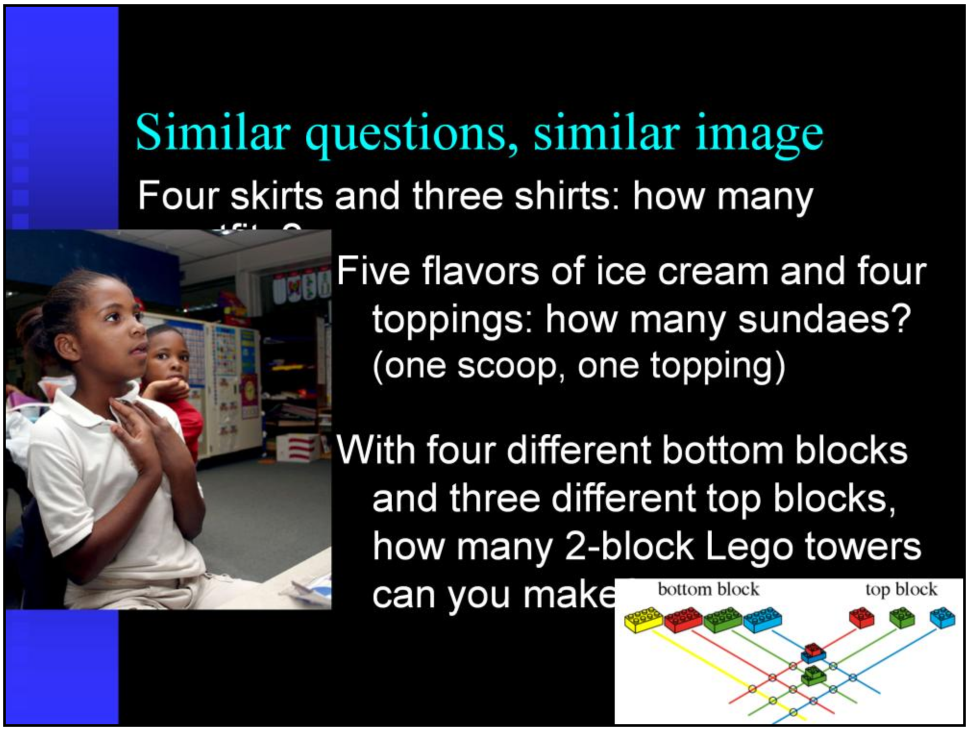


Seeing multiplication in combinations and intersections is often new for the teacher as well as for the child, an idea that is laden with many surprises.

As we can simplify the drawing of the marble-bag by erasing the top and bottom and leaving only the tied part (looking like the “x” of algebra), here, too, the *notation* -- the times sign -- is deliberately connected with an image that gives it meaning. The times sign looks like the street-avenue crossings.

The complexity of the word parts can be suited to the children’s abilities. Finding the non-words is fun for kids, excellent reading practice (check with the reading specialist!), and also fits with early mathematical goals of having children classify thoughtfully, looking for examples and non-examples of (whatever).

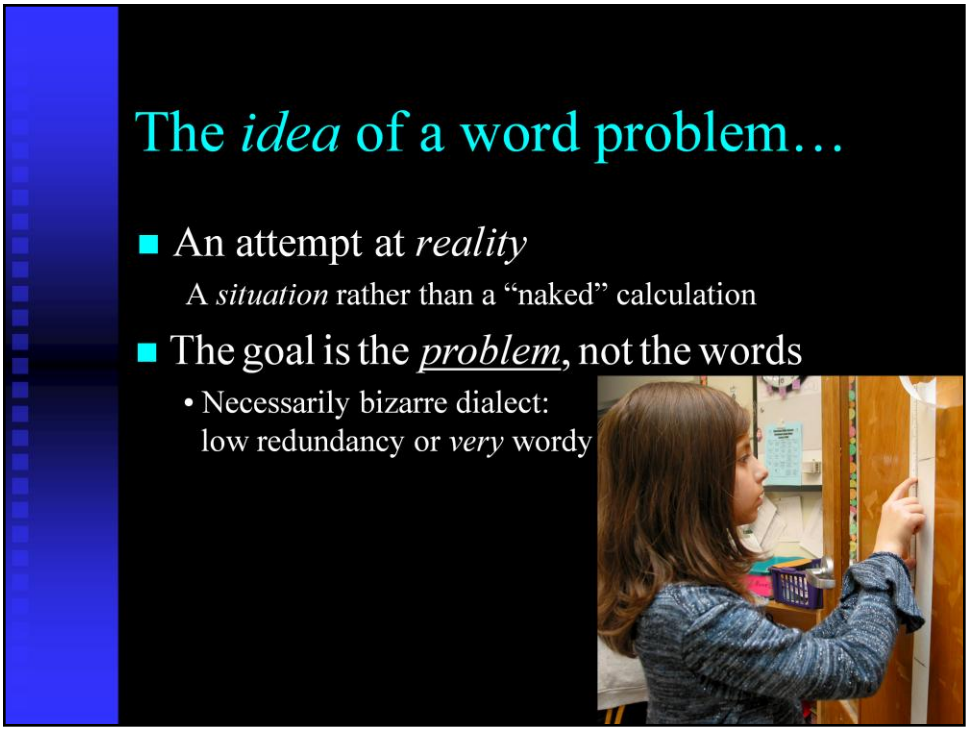
YOU CAN INVENT THESE COMBINATIONS ON YOUR OWN, BUT CHECK THEM OUT FIRST TO MAKE SURE THAT ALL THE “WORDS” THAT KIDS FIND ARE ONES YOU DON’T MIND THEM FINDING!



Making the kind of list that we needed in order to solve the previous puzzle involves a kind of systematic thinking that we start teaching very early (because it takes years to develop). It isn't mathematical "content," in the usual sense, but is an essential mathematical *skill* that is often overlooked (or blithely assumed to develop on its own). Problems like the ones shown here appear on tests (!). The real mathematical "meat" of them is in seeing how they connect to ideas of (1) multiplication, (2) coordinate systems, (3) "counting" without counting, (4) systematic enumeration, (5) thoughtful investigation.

Mathematical language; mathematical thinking

- Linguistics and mathematics
- Algebra as abbreviated speech (Algebra as a Second Language)
  - A number trick
  - "Pattern indicators"
  - Difference of squares
- Reading two dimensional information
- Word problems
- Developing mathematical language



As we've shown, problems need not be "word problems," but the word problem is an honest move, an attempt to convey, in a print medium, something like the nature of problems that "just happen to us" in "real life." (In older texts, this was more apparent, both to the modern reader and probably to the century-ago child.)

But the focus has become "learning how to do word problems" rather than "learning how to do problems" that, as it turns out, have been presented in words because we're in a print medium!

Unlike flowing prose, in which there is enough contextual redundancy and logical flow to make sense out of unfamiliar words or novel actions — people, even muggles, bring a lot of life knowledge to understanding the events in Harry Potter — word problems are short and terse. Every piece of information counts and is not repeated, unless the writer is deliberately being perverse (or is a lousy writer). This is a unique genre of writing that does not occur ANYWHERE but school, but it got that way naturally.

The *idea* of a word problem…

- An attempt at *reality*
  - A *situation* rather than a "naked" calculation
- "Clothing the naked" with words makes it *linguistically hard without improving the mathematics.*
- In tests, it is misleading, even discriminatory!
  - State ELA tests test ELA
  - State Math tests test Math and ELA

But the goal must remain THE PROBLEM, and if we could teach the problem solving without words, we'd be solving lots of *real* schooling problems.

Kids who just barely pass the ELA get sunk when they must use those same ELA skills AND math at the same time, but because they *are* passing ELA, their failures in math are attributed to the math, not to the double-whammy (ELA + math), and so the common conclusion is that they need more math drill. They get the message that they are terrible at math; their math experience is dummed down rather than spiced up (exactly the best remedy for many of them), and the spiral goes in the wrong direction.

Attempts to be efficient (spare)

- Stereotyped wording ⎝ key words
- Stereotyped structure ⎝ autopilot strategies

The unique genre *encourages* students to look for its characteristics. We DON”T want that result, but, in fact, this is *smart* behavior.

Key words

We rail against key word strategies.

> Ben and his sister were eating pretzels.
>
> Ben left 7 of his pretzels.
>
> His sister left 4 of hers.

So writers do cartwheels to subvert them.

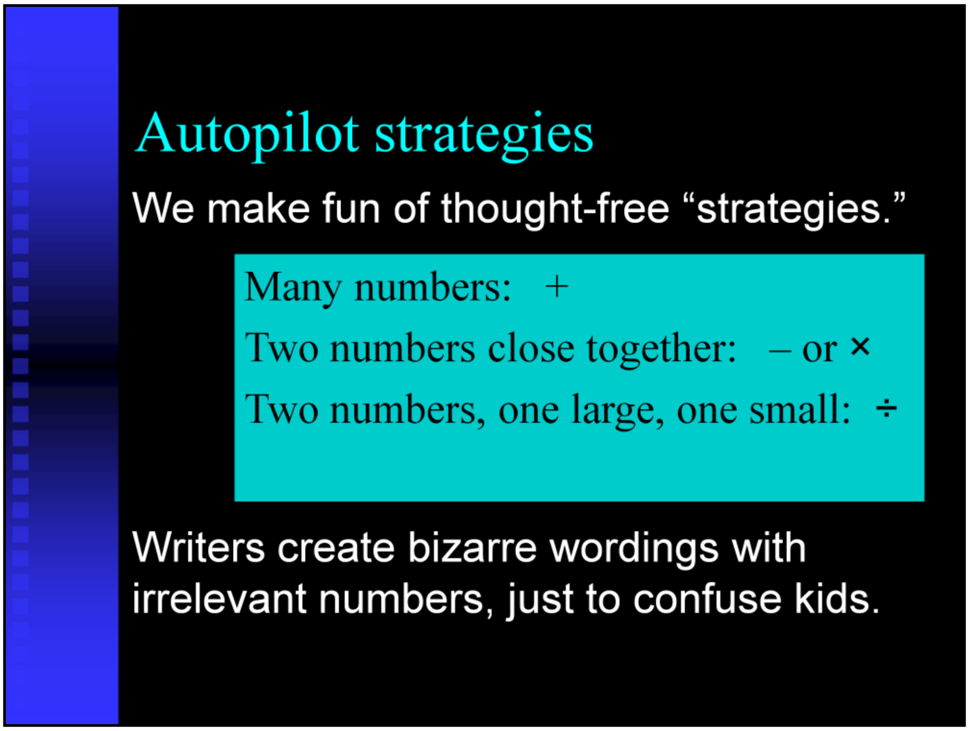
But, frankly, it is *smart* to look for clues!

This is how language *works*!

The obvious implication of "How many pretzels were left?" is SUBTRACT.

The writer's deliberate deception is, well, deliberate deception, not math.

Other key words like "altogether" for addition…



Good problems may, because of their math, be difficult, but taking simple arithmetic/mathematical tasks and making them deliberately confusing does not promote better mathematical learning. It simply becomes a contest between child and teacher/writer. Bad model!

If we want children to learn *mathematics* and to *think and communicate clearly*,

deliberately perverting *our* wording to make *our* writing unclear is not a good model!

So what *can* we do to help students read and interpret story-problems and communicate mathematically?

Mathematical language; mathematical thinking

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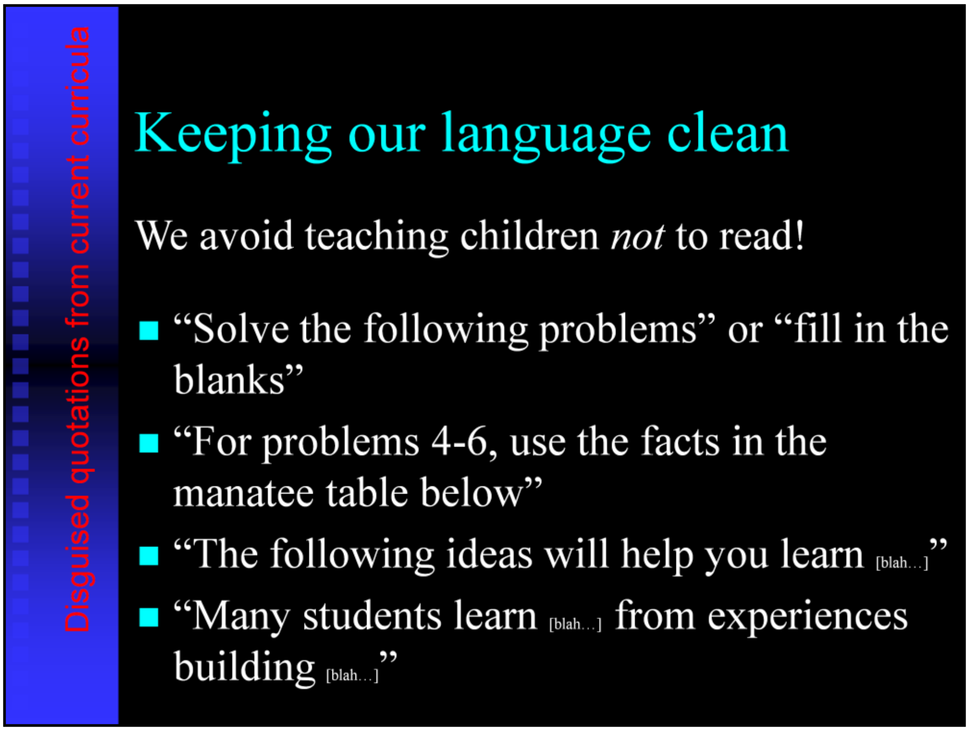


Each of these is a quote from a current math text for elementary school.

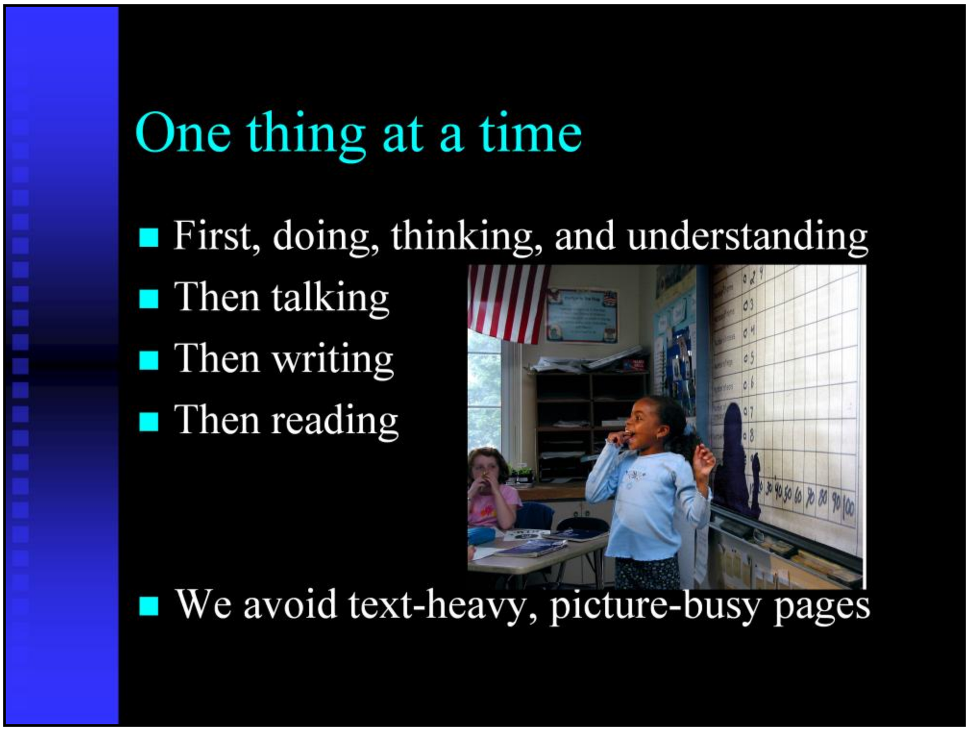
“Solve the following problems” teaches students not to read, because if they skip over it entirely, they’d still do the problems. It also seems to imply we think they’re stupid. Students *naturally* fill in blanks, sometimes even when they shouldn’t! To be literate mathematically, they need to recognize “where the action is” (e.g., blanks) and what is the appropriate action to take. In my opinion, we still have too much of this kind of thing—publishers need it for reviewers—but we have less than most texts.

“For problems 4-6, use the table below…” When we go to a restaurant, we read prices on a menu. We don’t expect a sign to say “prices are on the menu.” When a newspaper article has a table or graph, it may *refer* to data in it, but it doesn’t say “look at the table to understand this article” or even “look at the graph to see how much the cost grew.” Only when there is an ambiguity (e.g., multiple tables) should the text say which to look at. To be literate, students need to learn to *scan for information*, to *look for tables or graphs*, and to figure out what is the relevant information. When there are *many* problems on a page, there are still ways of making clear which ones go with which information without literally spelling it out. Spelling such things out treats students as non-thinkers. Again, we still have this occasionally, but much less than in most texts.

“The following ideas will help you learn…” Just wasted words. If they won’t help me learn, they shouldn’t be here. And besides, I want to do the learning, not hear about doing the learning. We have none of this at all.

“Many students learn…” And some don’t. Besides, I want to learn math, not psychology or pedagogy. We have none of this at all.

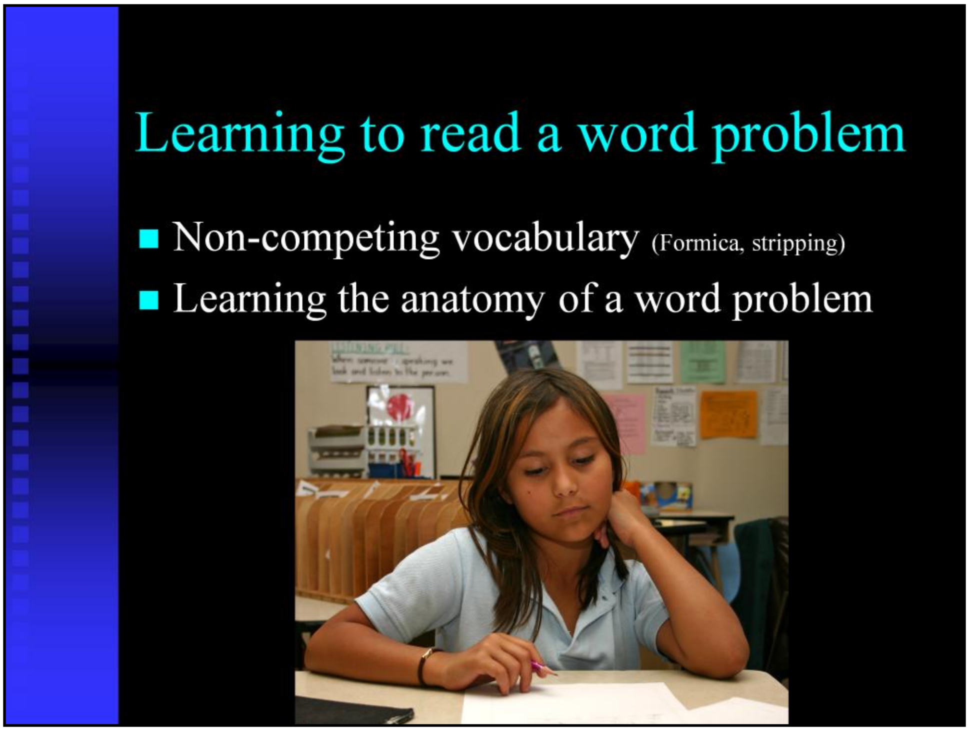
Kids are bored by this writing, and *should* be. It is not directed where *their* minds are.



Many pages in most elementary school math books have a *lot* of text on them.

It is important to teach students to be able to read mathematically, and many schools have instituted reading and writing across the curriculum, but doing two hard things at the same time is no favor to learning. Mathematics has invented its various special forms—symbols, formulas, tables, graphs—precisely because words are cumbersome for expressing these ideas.

Students will learn to read math story problems, and to write open-response answers for tests, but must first have the ideas—not just facts and procedures, but understanding and images—in mind, and to *show* they understand by *doing*. After doing, they learn to say. Then write.



Observed: a sixth-grade worksheet was using the context of building a table to present area and perimeter problems, and asked students how much Formica the builder would need for the surface of the table and how much stripping he would need. For most of the students, “stripping” was distracting and funny not just because of what it conjured up in their minds, but because that is the *only* meaning of the word that they knew. And, for many, Formica had no meaning at all. Because they are not sure whether unfamiliar terms are important to know or not, or even whether or not they are mathematical terms, this just stops them in their tracks.

The anatomy of a word problem: see next slide.

“Headline Stories”

- Less is more!

> Ben and his sister were eating pretzels.
>
> Ben left 7 of his pretzels.
>
> His sister left 4 of hers.

- What questions *can* we ask?
- Children learn the *anatomy* of problems by creating them—neonatal problem posing!

One solution. Take any “word problem” and omit the question.

Then ask what questions CAN be asked of this information.

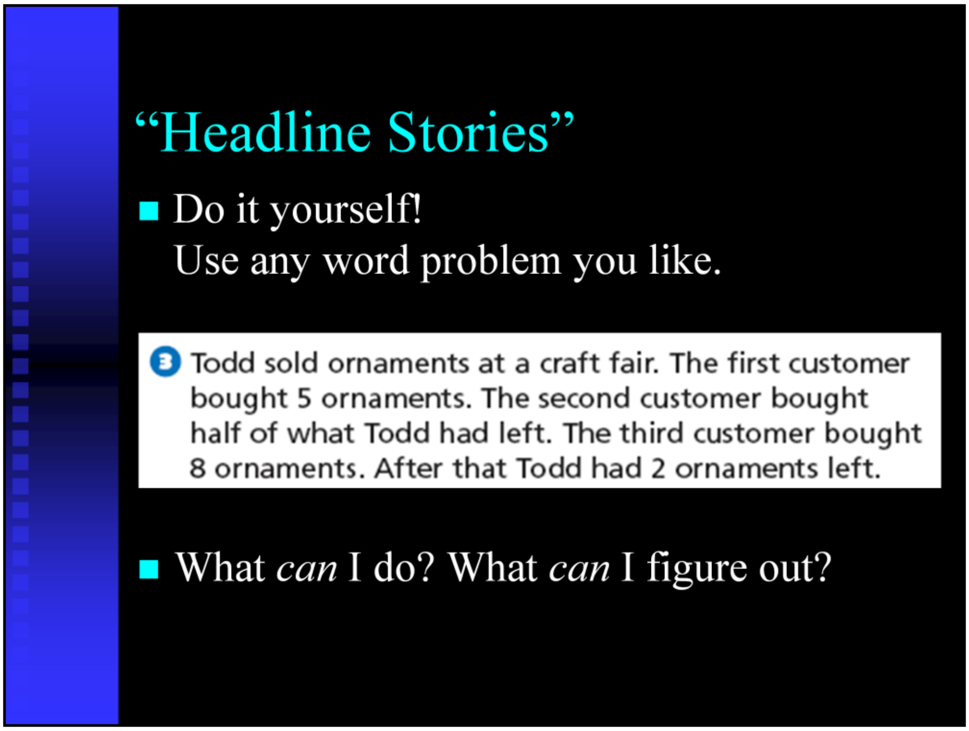
Who left more? How many more did Ben leave? Could they have started with the same number of pretzels? Could Ben have started with 6 pretzels? How many, altogether (yes, the key word), were left? How many pretzels did Ben eat? Right, we can’t answer that last one, but we CAN ask it. And what information would we need in order to answer it?

And what information might we like to ADD to this problem?

If Ben started with 5 more pretzels than his sister, what can we say? (ANYTHING?)

Let the kids build up possible word problems, learning the structure of word problems by experiencing, first hand, how they are constructed.

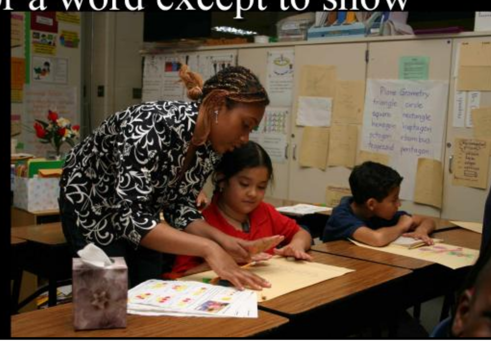
For more about this approach, see http://thinkmath.edc.org/index.php/Headline_Stories.



In general, the approach should not be "what *should* I do?" but "what *can* I do?"

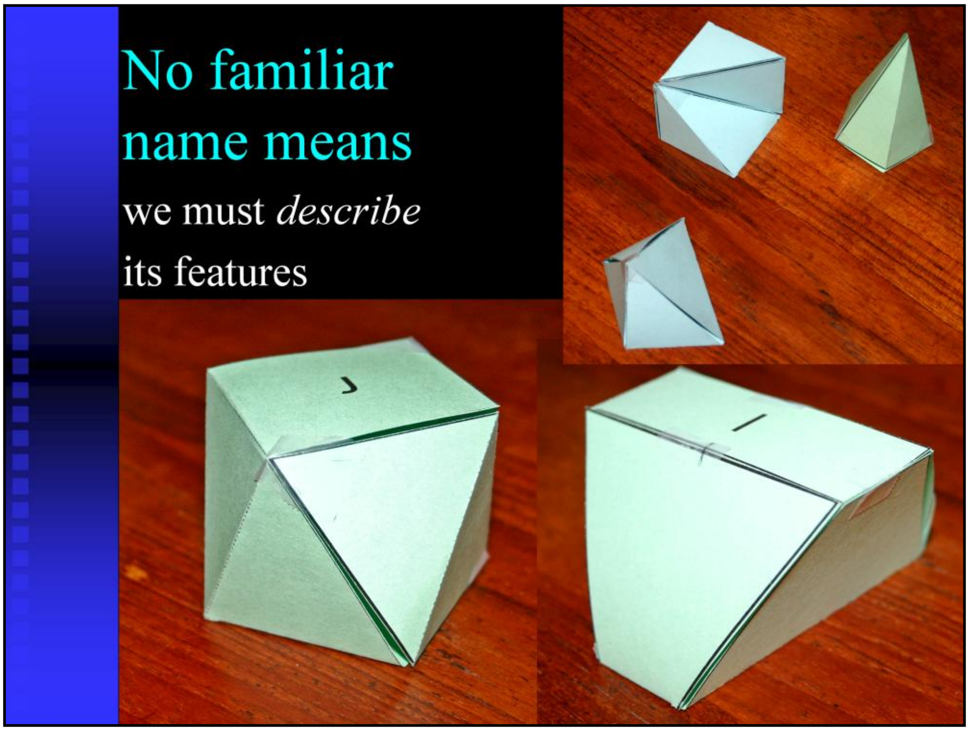
Acquiring mathematical vocabulary

- Need some *communicative* context for it.
- Geometry is full of words that kids can't use *communicatively*: trapezoid, prism, vertex.
- Having no use for a word except to show that you know it is *not enough.*

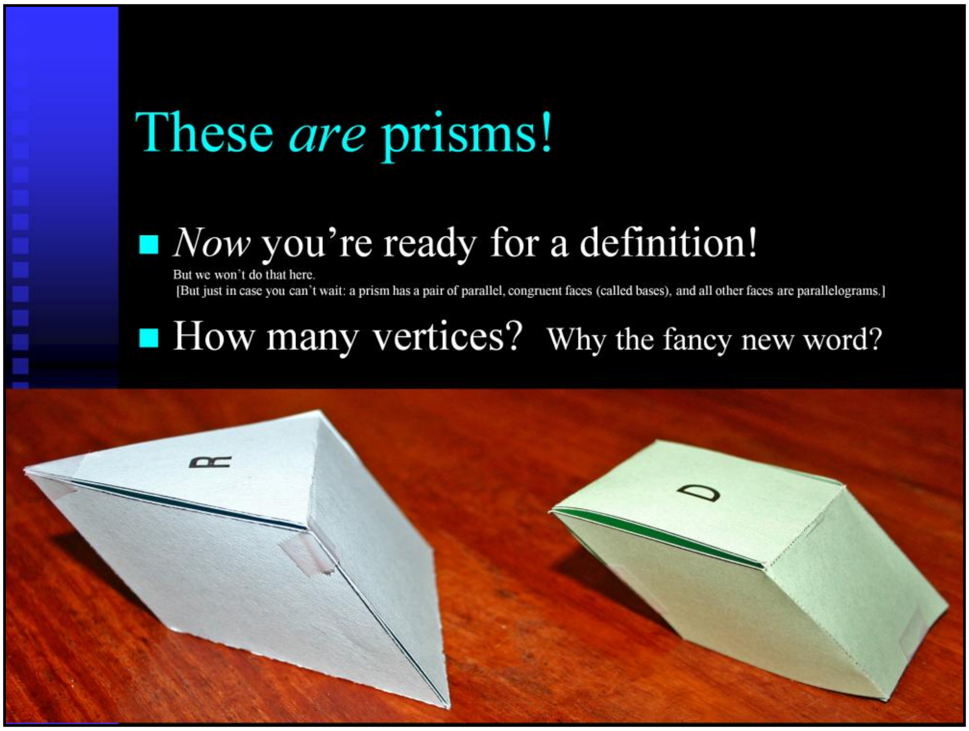


Saying "this is a prism and that is not a prism" or "a prism has two bases" is not a *communicative* use of the word. It merely shows that we know what a prism is, but doesn't help us say anything we could not have said without that word.

The yellow block in front, labeled B, is perhaps the most standard example of a “prism.” In fact, almost all of the figures in the foreground of this picture are prisms! Prisms can “lean to the side” (like E and S near the front); they can have triangular (S) or pentagonal (green, near the standing L) or trapezoidal bases, or even be L-shaped (yellow) standing with one of its bases facing us. There are also some pyramids way in the distance. And some creatures (like the stacked red ones on the far left next to S, and the green one just behind B) that are neither prisms nor pyramids.



Shape I has six faces. Its top and bottom face are congruent oblong rectangles. The other four faces are trapezoids and they look like they're congruent, too. Shape J has two square faces and eight triangular faces. It's also the only shape we have that has more faces than vertices. Every other shape has either more vertices than faces, or are pyramids, which have the same number of vertices as faces.



After a bunch of experience with prisms and non-prisms, a definition could make sense, but any definition before describing the great variety of prisms would be misunderstood.

What can we *say* about prisms? We can look for patterns that characterize prisms. Here's one.

How many vertices on the triangular prism? Well, three are touching the table, and three are around the top, so 6. How many on the square prism? Four touching the table, and four around the top, so 8. Can we see some general principle yet (not just a pattern, but a reason for the pattern) or do we need more data?

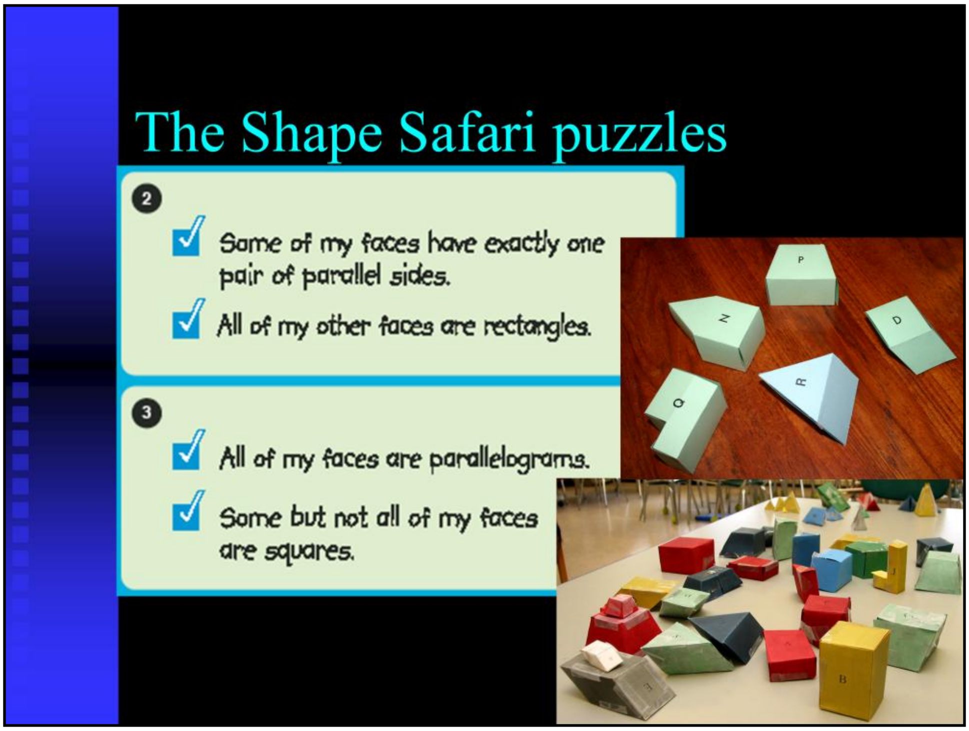
And how many faces on the triangular prism. Well, one "side face" for each side of the floor triangle, so 3 side faces, and then the top and bottom. What about the square prism? Well 4 side faces and then the top and bottom. Any general principle or do we need more data?

In both, we have more vertices than faces. This is only two cases, so maybe we can't generalize yet! But maybe we *can* come up with some way of thinking about it to decide whether that's likely to happen *only* these two times, a few times more, often, or always.

By the way, why do we need such fancy words as "faces" and "vertices"? What's wrong with "side" and "corner"? The problem with those words is that they're ambiguous. If I tell you to stand in the corner of the (rectangular) room, you are likely to find the place where two walls meet. If I ask you how many corners there are, you might well say four. But, for a spider, the room has eight corners! Eight places where two walls and a floor or ceiling meet. So "corner" is not quite clear any more. There is nothing wrong with saying that a triangle has three corners -- it may not be "grown up" vocabulary, but it is not ambiguous. But in 3-D, "corner" *IS* ambiguous, so we don't use it. The places (line segments) where two faces meet are called "edges." The places (single points) where three or more faces meet are called "vertices." We use new words not just because "that's how it is said" but because they help us communicate clearly.

In the same way, when I ask you how many "sides" a square (like the top of the green prism) has, and you say 4, you are referring to the line segments that surround that square. If I then ask you how many "sides" that green prism has, how can you tell what I mean? Am I still talking about line segments (in which case the answer is 12) or about 2-D shapes that make up the walls, ceiling, and floor of that leaning room? Again, we choose a new word -- face -- to refer clearly, and only to walls, ceiling, and floor.

[Note to presenter: I don't bring this up in presenting because it is too technical, but people sometimes ask, so I include it in the notes. I've avoided using the word "base." That's also a very useful term, but it's miserably difficult to get clear. Like all words, it helps to know not only what it *is*, but also what it is not. It is *not* "whatever face the object happens to be sitting on at the moment." In slide 20, the "house" and "chair" shaped prisms (N and Q) are standing so that they look like a house and a chair. In casual English, the word "base" means "whatever fact the object would normally sit on," (like the base of a lamp or the basement of a building), but mathematics uses the word "base" in the sense of "the *basis* for this construction, "the polygon that the 3-D structure is based on." R is clearly "based on" a triangle. With any prism whose faces are all some kind of parallelogram (like D), *any* face could be the basis, so what the "base" is can't be defined neatly. The way D is set in this picture makes it appear to be "based on" a square, but it could actually be "based on" any of its faces. A similar ambiguity occurs with pyramids. If one face is not a triangle, and all the others are, then the non-triangle is the face that the pyramid "is based on" and so it is the base. If *all* of the faces are triangles, then any one of them *could* be the base (and the pyramid might lean or otherwise look as if it has been knocked over, but life is just like that sometimes!). In teaching, use base *in context* and avoid technicalities, because *definition* is a horrible mess, and we either wind up defining it wrong or, in an effort to get it right, make the definition so complex that nobody understands anyway.]

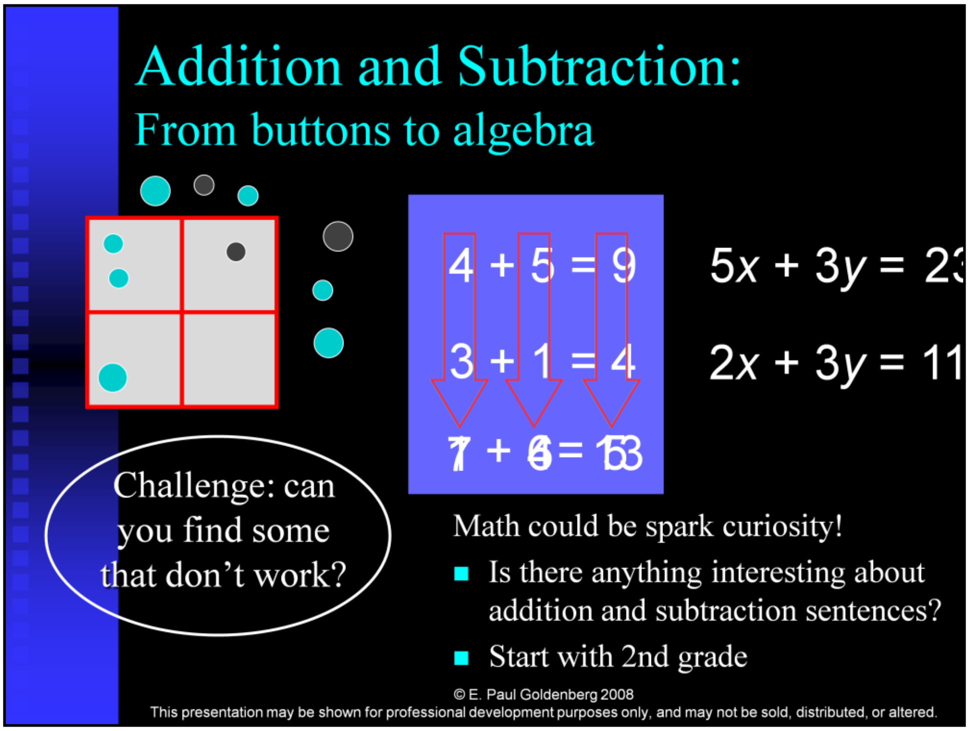


Why puzzles?

Kids feel really victorious (smart) when they solve these puzzles. Why? Probably because we've evolved to feel that way! Our victories are not with teeth or claws or swiftness, but with brains, so the triumph of the brain gives us that victorious feeling.

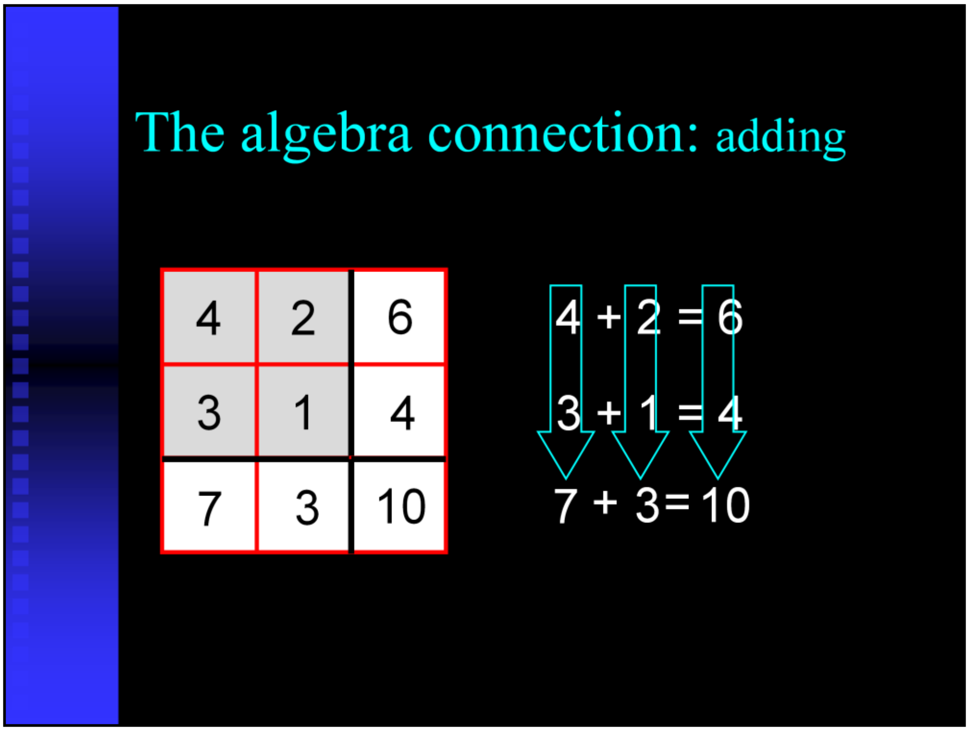
Cats play/practice pouncing, and they sharpen their claws by scratching. We play/practice by using our brains, and our neural pleasure center says "yeah, do that more" because, for our species, sharpening our wits means survival. If we need more evidence to believe that, consider that Sudoku and cross-word and word-search puzzles have absolutely no "practical, real world" utility, but are so popular that they sell in *super-markets.* People have different tastes in challenge -- and some get scared out of expressing that taste or fearful that they will fail -- but we are build, from the start, to like challenges that we can meet.

And why *these* puzzles? Because they engage the intellect and *also* get kids *talking*, using the language they need.

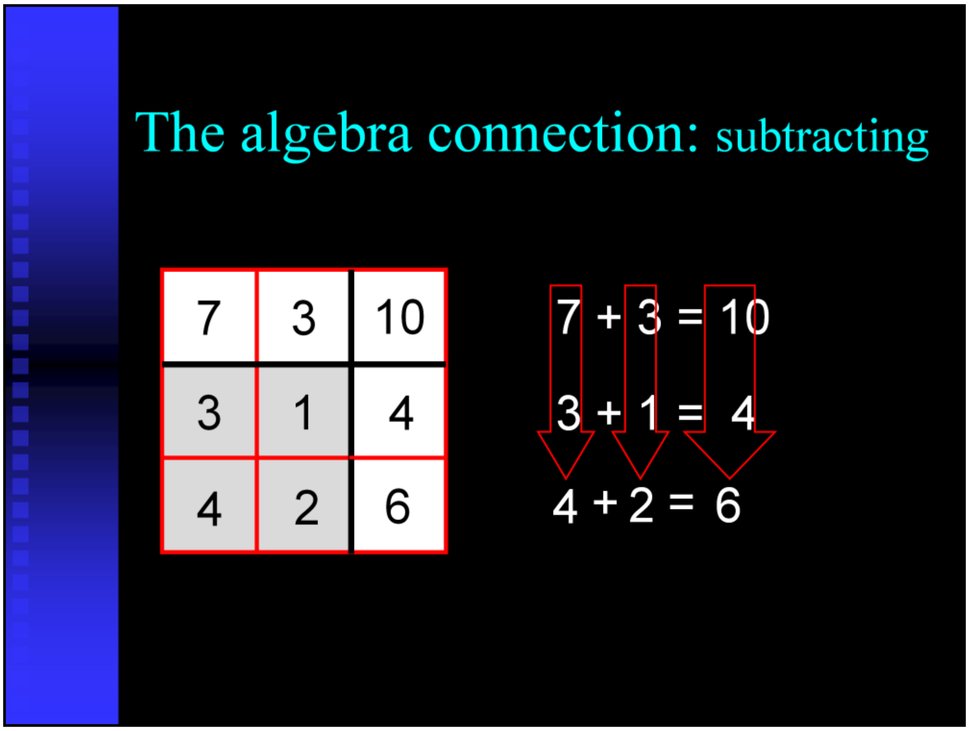


For more about the connection between kindergarten sorting and solving simultaneous equations (the bridge being the adding or subtracting of number sentences), see http://thinkmath.edc.org/index.php/Cross_number_puzzles

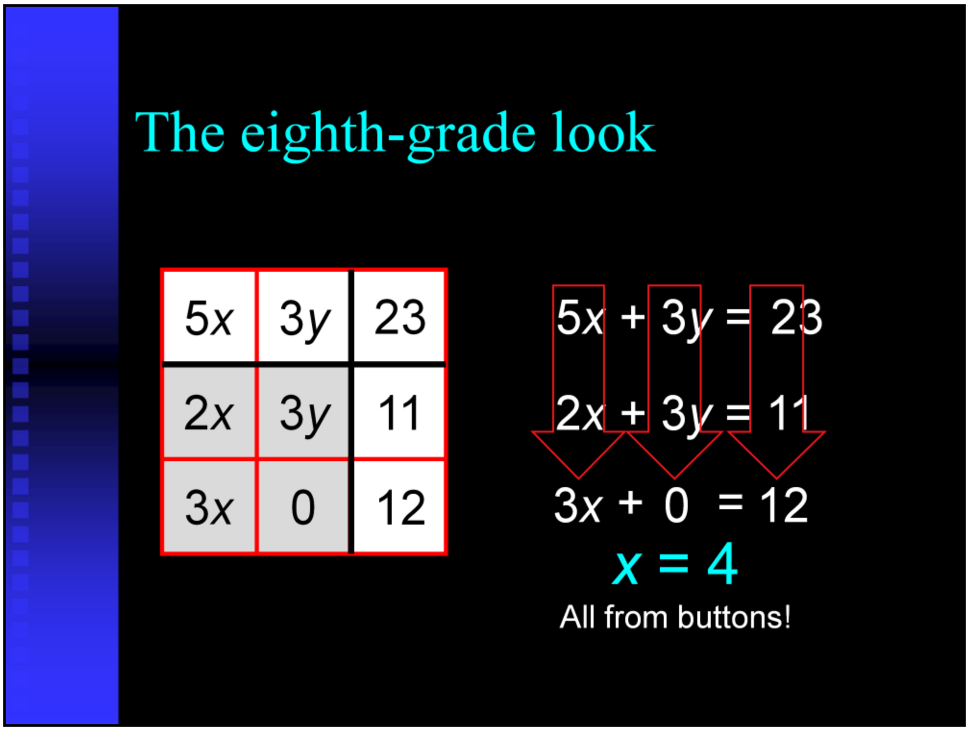
Kids *not* troubled by missing addends!



Not only can we add addition sentences to get a new addition sentences, but we can also add subtraction sentences to get a new subtraction sentences.



And can subtract subtraction statements… In fact, once negative numbers are meaningful to kids, one can even “subtract a sentence” like 4 + 5 = 9 from the 7 + 3 = 10. The resulting numbers, in order would be 3, -2, and 1, which can be assembled into the sentence 3 + -2 = 1 (or 3 – 2 = 1).



Well, these are numbers, too! The first line says “Five bags of one size plus three bags of (maybe) another size contain 23 marbles.” So we can use the logic of the cross number puzzle -- the logic of the buttons! -- to create a new sentence. In this case, it looks like subtracting one sentence from the other is the best idea.