Mathematical Proficiency: An Example from the Inquiry Oriented Differential Equations Project

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Mathematical Proficiency (NRC, 2001)

Mathematical proficiency has five strands:

- **conceptual understanding**—comprehension of mathematical concepts, operations, and relations
- **procedural fluency**—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- **strategic competence**—ability to formulate, represent, and solve mathematical problems
- **adaptive reasoning**—capacity for logical thought, reflection, explanation, and justification
- **productive disposition**—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy.

How to positively effect students’ mathematical proficiency? Some results and insights from the Inquiry Oriented Differential Equations project.
Inquiry-Oriented Differential Equations

Background theory

- Dynamical systems point of view
- Instructional design theory of Realistic Mathematics Education
- Java applets
- Co-emergent analytic, numerical, and graphical approaches
- Systematic study of student thinking
- Social Negotiation of Meaning
- Role of the teacher

Innovative curriculum
Innovative technological tools
Innovative pedagogy
Transmission of knowledge Approach
An Inquiry-Oriented Approach

**Student activity**
- Students routinely explain and justify their thinking, listen to and attempt to make sense of others’ ideas.
- Students engage in genuine argumentation as they (re)invent mathematical ideas.

**Teacher activity**
- Teachers routinely inquire into how it is that students are thinking about the mathematics.
- Teachers are continually attempting to understand their students’ mathematical reasoning.
Systematic investigations of student learning

Quantitative methods (Statistical analysis)

- Individual interview
  - to probe student understanding in depth
- Written questions (pretests and post-tests)
  - to ascertain prevalence of specific difficulties
  - to assess effectiveness of instruction
- Descriptive and explanatory studies during instruction
  - to provide insights to guide instructional design

Qualitative methods (Interpretative analysis)
In the IO-DE project

• Students reinvent many key mathematical ideas (for example, Euler’s method, bifurcation diagrams, analytic technique for solving systems of linear differential equations)

• So what are the benefits for learners? Do they learn ideas more deeply? Do they retain their knowledge longer? Do they develop more productive dispositions about mathematics, about themselves, and their ability to do mathematics?
Assessment of students’ conceptual understanding and procedural fluency

- 4 different sites, N = 111
Students’ retention of conceptual understanding and procedural fluency

Mean Scores

QG: Qualitatively/Graphically
M: Modeling
PO: Procedurally Oriented

IO-DE(n=15) TRAD-DE(n=20)
Influencing student beliefs and values

• One way to give explicit attention to promoting productive dispositions and habits of mind in the mathematics classroom is for the teacher to be deliberate about initiating and sustaining particular classroom norms
• Some theoretical background on the relationship between classroom norms and beliefs
• Illustrative examples that demonstrate how this theoretical framing can be used to explain change in a student’s beliefs and values
Beliefs can be thought of an individual’s understandings of normative expectancies. Social norms can be thought of as taken-as-shared beliefs that constitute a basis for communication.

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Initial Beliefs

J1 - The only thing that I find sort of confusing is the fact that there may not be an exact answer or answers to a specific problem. There are so many variables in some of these problems that the answers that are obtained (if obtained) can only be used to make educated guesses. I'm used to thinking of math as an "exact science" where there is always an exact answer or answers to a problem. Indicates the student’s belief that mathematics should always yield one exact answer was being challenged.

J1 - I’m still getting used to the format. I’m more used to the teacher saying everything and not letting the students really have a "voice." Belief about the teacher’s and the students’ roles

J2 - [M]ost of the points lost were due to my failure to explain how I reached my answers. I thought a clear, systematic approach to the math calculations would be sufficient to explain my thought process. I now have a better understanding of the expectations. This student is articulating that he formerly believed that in mathematics explanation is not required.

P1 - I like the way the class and the book concentrate on practical applications and explanations for differential equations. As you may have noticed from my info card, I have taken this class before at ______ [another campus of the same university] and it was very different. We spent a lot of time trying to memorize all the techniques to solve the equations and learned very few practical ideas. I was lost and disinterested 15 minutes into the first class session. I can honestly say I think I’ve learned more about differential equations in the first two weeks here than I did in the whole semester there. This student explicitly linked expectations of memorizing techniques to not learning much about differential equations and expectations of explanation to substantive learning.
Negotiating Classroom Social Norms

Excerpts from a twenty-minute episode on day 2

_Instructor:_ Just to sort of recap, last time we were dealing with the spread of a virus. We had [the] elementary school population in Chicago where we had students who were either susceptible to the disease, who were recovered, or who were infected. And we talked about one differential equation. That was \( \frac{dR}{dt} = \frac{1}{14}I \). Anyone remember why it was one fourteenth? How many people remember? Shawn, why was it \( \frac{1}{14} \)?

_Shawn:_ Fourteen days from the time you got cured, from the time you got it to the time it's over.

_Shawn_ likely expected the instructor to evaluate his response as correct or incorrect and then initiate a different question. Instead, however, he calls for additional explanation, in particular, he asks how it “makes sense.” In doing so, he indicates that students’ responses should explain their individual thinking and further, that mathematical thinking is about sense-making.

_Instructor:_ Okay, can you explain to us then why it was \( \frac{1}{14} \) times \( I \)? How did that sort of make sense as a way to express the change in the recovered population?

_Shawn:_ That it's constant.

_Instructor:_ Say that again.

_Shawn:_ That it's constant. The same amount of number of people for each stage.

With this last question the instructor initiates another shift. His question indicates that he expects others to be actively engaged in the discussion; they are to listen to the exchange he and Shawn are having and are to develop their own interpretations about Shawn’s response.
Dave: The way I thought about it at first, to make me think that all the points weren’t saddles, is that if the next one was a saddle—see how [Bill] has got the one line coming in towards [referring to the phase portrait that Bill had drawn on the blackboard]. Well, if the next one was like that, then you would have to have another point in between those two equilibrium points, like separating, like a source or something. So that’s how I started thinking about it. So then 3π/2 might be a source or maybe a saddle point with opposite direction.

Bill: So it’s like, you’re saying that if there is a saddle, there has to be a source. If there is a sink or a saddle you have to have a, like in this case right here, you would have to have a source in between the saddles in order for it to really make sense.

Shortly thereafter, Bill relates Dave’s explanation to their earlier study of autonomous first order differential equations.

Bill: If you draw the phase line with, like, two sinks, one on top of the other, then you would have to have a source between them.

These spontaneous remarks made by Dave and Bill indicate that they have taken seriously the obligations of developing personally-meaningful solutions, of listening to and attempting to make sense of the thinking of others, and of offering explanations and justifications of their mathematical thinking. In the process of acting in accordance with these expectations they are demonstrating their beliefs about their roles and about the nature of classroom mathematical activity.
Coordinating Sociomathematical Norms (what counts as an acceptable explanation) and Specifically Mathematical Beliefs

Two aspects of acceptability: (1) when it serves a clarifying function and (2) Grounded in an interpretation of rate of change (versus in terms of procedures).

There is considerable evidence that by the fourth electronic journal assignment the students were beginning to understand the clarifying function of explanation. Many students wrote comments to the effect that their explanations were inadequate. For example, one student ended his journal with this comment, “This is the best way I know to explain it which I know is very lacking.” Another wrote, “I am not completely sure I understand this point so I wouldn’t try to explain it to someone unless they had some feedback as to what they think it is.”

*Instructor:* Okay, so Jerry says that if the population gets above 8 they [the fox squirrels] are going to start dying. Tell us why you made that conclusion.

*Jerry:* Because some number greater than 8 over 8 is going to yield some number greater than one, which 1 minus something greater than 1 is going to give you a negative number and so something times a negative number is going to give you a negative number, so your slope is going to be negative.

*Instructor:* And so what does that mean for us? That means what? If this term is negative, that doesn’t tell us anything in itself in relation to the differential equation.

These comments are indications of the students’ belief that the purpose of explanations is to communicate; explanations should clarify one’s thinking for others. Relates to the belief that mathematical explanations have to be grounded in ideas rather than procedures.
Concluding Remarks

• Theoretical comment
  By coordinating individual and collective perspectives, we give primacy neither to the social nor the psychological. Rather, we maintain that each provides a backdrop against which to consider the other.

• Pragmatic comment
  Verschaffel, Greer, and De Corte (1999) have noted that it is generally assumed that students’ beliefs about mathematical activity develop “implicitly, gradually, and tacitly through being immersed in the culture of the mathematics classroom” (p. 142). While one generally may agree with this statement, I argue that one way to give explicit attention to student beliefs in the mathematics classroom is to be deliberate about initiating the negotiation of classroom norms. While norms are not rules set out in advance, the teacher is in a unique position to influence the evolution and nature of classroom norms, and hence positively effect student beliefs and values.
IODE References


