Habits of Mind for Proving

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Observations from a Design Experiment

- A one-semester, 3-credit course for prospective and beginning math graduate students.
- Meets 2 times/week for 75 min.
- Purpose is to teach proof construction
- Significant modification of the R. L. Moore Method (Jones, 1977; Mahavier, 1999)

- Notes with statements of theorems, definitions, requests for examples, but no proofs and only minimal explanations

- No lectures

- Students work outside of class and present their proofs at the blackboard

- We read and check each proof, “thinking aloud” so students can see what we are checking.

- We offer, sometimes extensive, criticism and advice
- Halfway through the course, we have students validate each other's proofs
- Course has practical value because professors assess students' understanding by asking them to prove theorems
- We have just completed our 3rd iteration of the course

- Everything is video recorded.
- Field notes are taken.
- This information is analyzed in planning sessions between class meetings in order to influence students' learning trajectories.

- We distinguish two aspects or parts of proofs: the *formal-rhetorical* and the *problem-oriented* parts.

- The *formal-rhetorical* part of a proof is the part that one can write based only on logic, definitions, and sometimes theorems, without recourse to conceptual understanding, intuition, or genuine problem solving.

- We call the remainder of the proof the *problem-oriented* part and it does require conceptual understanding and genuine problem solving.

(Selden & Selden, in press)

- We first concentrate on having students write the formal-rhetorical parts of proofs.

- Doing this, exposes the “real problem.”

- In the proving process, actions are responses to (inner) situations.

- After similar situations occur in several proof constructions with the same resulting action, a link is built between the situation and the action.

Example

- In a situation calling for *C* to be proved from *A* or *B*, one constructs 2 independent subproofs arriving at *C*, one supposing *A*, the other supposing *B*.

- If one has had repeated experience with such proofs, one does not have to think about doing or justifying this action, one just does it.

- We call such persistent (small grain-size) linked situation-action pairs, *behavioral schemas*.

- We see behavioral schemas as a form of (often tacit) procedural knowledge that yields immediate (mental or physical) actions. They call for knowing *how* to act. They are similar to what Mason & Spence (1999) have called “knowing *to* act in the moment.”

- Within a broad context, such schemas are always available – they do not have to be searched for or recalled.

- Taking a more external, or third person, view and perhaps a larger grain-size, behavioral schemas may also be seen as *habits of mind*. (Margolis, 1993)

- We want to encourage good habits of mind and discourage detrimental ones.

Several examples of how and we try to encourage helpful behavioral schema and discourage unhelpful ones

- ***Focusing too soon on the hypotheses***

Moore (1994) described undergrad transition-to-proof students who could not prove on the final exam: If f and g are functions from A to A and

$f \circ g$ is 1-1, then g is 1-1. He said that students started in the wrong place, with the hypothesis, instead of supposing

$g(x) = g(y)$.

- Like (Bob) Moore, we have found that a number of our students habitually focus on the hypothesis immediately, instead of unpacking the conclusion and trying to prove that.

- By patiently guiding students to first write the formal-rhetorical parts of proofs, this detrimental schema, or habit, can be overcome.

- ***Proving universally quantified statements***

One often starts the proof of a statement “For all (numbers) x $P(x)$” by writing “Let x be a number,” meaning x is “fixed, but arbitrary.”

Some students are reluctant to write this in their arguments.

Students eventually come to do this as if they were enacting a behavioral schema.

Illustrative Example

Mary was a returning grad student taking beginning real analysis with Dr. K, who assigned 3 or 4 weekly proofs, graded them very thoroughly, and allowed them to be resubmitted. He emphasized things like writing "let x be a number" into proofs.

Mary recalls feeling this requirement was not particularly important or appropriate. She did so to get full credit.

Near the middle of the course, Mary came to feel that writing things like “let x be a number” into proofs “made sense and it was the way to do it.”

She reports now, two years later, that she cannot think of any other way to write (this aspect of) proofs.

For Mary, this positive behavioral schema took long to develop, but has now become a well-developed habit of mind.

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- ***Showing an object is in a set***

Here is an example of a tutor, leading a student, Sofia, towards constructing a behavioral schema.

This occurred in the middle of the Spring 2008 course, and was devoted to helping Sofia prove Theorem 20: *Let $(X, \boldsymbol{U})$ be a topological space and Y a subset of X. Then $(Y, \{U \cap Y \mid U \,\varepsilon\, \boldsymbol{U}\})$ is a topological space (called the relative topology on Y).*

Sofia said she didn't know how to prove the theorem. At the tutor's suggestions she wrote the first and last lines (the formal-rhetorical part) and drew a sketch. With guidance, she unpacked what was to be proved into 4 parts (the defining properties of a topology). She proved Y is in the relative topology, but could not prove the empty set was in the relative topology. It became clear she did not know how to show an object is in a set, when the defining variable in the set is compound (for example, $U \cap Y$).

The tutor forgot about the theorem for a moment and asked Sofia if 6 is an element of $\{2n \mid n \varepsilon \mathbf{N}\}$ and why. She said yes, because 6 = 2 x 3 and

$3 \varepsilon \mathbf{N}$.

Using this as a model, Sofia was able to show the empty set was an element of $\{U \cap Y \mid U \varepsilon \boldsymbol{U}\}$ and do the third and fourth parts of the proof.

The tutor’s guidance facilitated Sofia’s construction of a behavioral schema (habit of mind) in which the situation is needing to show an object is in a set (where the defining variable is compound), and the action is showing the defining property is satisfied.

- ***Proofs requiring a previous result***

We have begun distinguishing 3 kinds of proofs, beyond those following immediately from definitions:

1. Those requiring a result in the notes.
2. Those requiring a result *not* in the notes, but easily articulated and proved.
3. Those requiring a result *not* in the notes that is *not* easily articulated and proved.

We try to provide experiences with all three of these, as we want "looking back" to become a habit.

For example, this occurred in Fall 2007 for Theorem 24 that states polynomials are continuous. Theorems 19-23 stated that sums and products of continuous functions are continuous and that the identity and constant functions are continuous. This is enough to prove Theorem 24 by induction, but there were students who did not notice this and could not prove it.

For an example of a Type 3 proof, we turn to the Fall 2007 notes where a semigroup is defined to be a nonempty set S together with an associative binary operation, and an ideal I of S to be a nonempty set so that $IS \cup SI$ is contained in I.

Theorem 46 states that *if S is a commutative semigroup with no proper ideals than S is a group.*

The first result needed is that if $a \varepsilon S$, then aS is an ideal (and hence $aS = S$).

The second result needed is that if $aS = S$, then the equation $ax = b$ can be solved for x.

These results were *not* in the notes in order that students could experience a Type 3 proof.