A COLLECTION OF LISTS OF MATHEMATICAL HABITS OF MIND

Kien Lim
University of Texas at El Paso
kienlim@utep.edu

Characteristics of general habits of minds (Cuoco, Goldenberg, & Mark, 1996)

- Pattern sniffers
- Experimenters
- Describers
- Thinkerers
- Inventors
- Visualizers
- Conjecturers
- Guessers

Habits of mind specific to mathematics (Cuoco, Goldenberg, & Mark, 1996)

- Talk big and think small
- Talk small and think big
- Use functions
- Use multiple points of view
- Mix deduction and experiment
- Push the language
- Use intellectual chants
- Algebraic approaches to things
    a. like a good calculation
    b. use abstraction
    c. use algorithms
    d. break things into parts
    e. extend things
    f. represent things
- Geometric approaches to things
    a. use proportional reasoning
    b. use several languages at once
    c. use one language for everything
    d. love systems
    e. worry about things that change
    f. worry about things that do not change
    g. love shapes

Habits of Mind for Arithmetic-Algebraic Transition (Mark, Cuoco, Goldenberg & Sword, 2010)

- Abstracting regularity from calculations
- Articulating a generalization using mathematical language

Mathematical Habits of Mind for Young Children (Goldenberg, E. P., Shteingold, N., & Feurzeig, 2003)

- Thinking about word meaning
- Justifying claims and proving conjectures
- Distinguishing between agreement and logical necessity
- Analyzing answers, problems, and methods
- Seeking and using heuristics to solve problems

Mathematical Habits of Mind for Secondary Students (Levasseur & Cuoco, 2003)

- Guessing
- Challenging solutions, even correct ones
- Looking for patterns
- Conserving memory
- Seeking special cases
- Using alternative representations
- Classifying carefully
- Thinking algebraically

General Habits of Mind for High School Mathematics (Cuoco, Goldenberg, & Mark, 2010)

- Performing thought experiments
- Finding, articulating, and explaining patterns
- Creating and using representations
- Generalizing from examples
- Articulating generality in precise language
- Extracting mathematics to make sense

Analytic and Geometric Habits of Mind for High School Mathematics (Cuoco, Goldenberg, & Mark, 2010)

- Reasoning by continuity
- Seeking geometric invariants
- Looking at extreme cases and passing to the limit
- Modeling geometric phenomena with continuous functions

Algebraic Habits of Mind for High School Mathematics (Cuoco, Goldenberg, & Mark, 2010)

- Seeking regularity in repeated calculations
- Delayed evaluation—seeking form in calculations
- Chunking—changing variables to hide complexity
- Reasoning about and picturing calculations and operations
- Extending calculations to preserve rules for calculating
- Purposefully transforming and interpreting expressions
- Seeking and specifying structural similarities

Algebraic Habits of Mind (Driscoll, 1999, 2001)

- Doing-undoing
    a. Working the steps of a rule or procedure backward
    b. Finding input from output, or initial conditions from a solution
- Building rules to represent functions
    a. Organizing information in ways useful for uncovering patterns and the rules that define the patterns
    b. Noticing a rule at work and trying to predict how it works
    c. Looking for repeating chucks in information that reveal how a pattern works
    d. Describing the steps of a rule without using specific inputs
    e. Wondering what different information about a situation or problem may be given by different representations, then trying the different representations
    f. Describing change in a process or relationship
    g. Justifying why a rule works for "any number"
- Abstracting from computations
    a. Looking for shortcuts in computation, based on an understanding of how operations work
    b. Thinking about calculations independently of the particular numbers used
    c. Going beyond a few examples to create generalized expressions, describe sets of numbers, or either state or conjecture the conditions under which particular mathematical statements are valid
    d. Recognizing equivalence between expressions
    e. Expressing generalizations about operations symbolically
    f. Using generalizations about operations to justify computational shortcuts

Geometric Habits of Mind (Driscoll, DiMatteo, Nikula, & Egan, 2007)

- Reasoning with relationships
- Generalizing geometric ideas
- Investigating invariants
- Sustaining reasoned exploration by trying different approaches and stepping back to reflect while solving a problem

Habits and Values of Mathematicians (Seaman & Szydlik, 2007, p. 170-171)

- Seek to understand patterns based on underlying structure
- Make analogies by finding the same essential structure in seemingly different mathematical objects
- Make and test conjectures about mathematical objects and structures
- Create mental (and physical) models for examples (and non-examples) of math objects
- Value precise mathematical definitions of objects
- Value an understanding of why relationships make sense
- Value logical arguments and counterexamples as our sources of conviction
- Value precise language and have fine distinctions about language
- Value symbolic representations of, and notation for, objects and ideas

Four Groups of Reasoning Habits in NCTM's *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM, 2009, p. 9-10)

1. Analyzing a problem
    - Identifying relevant mathematical concepts, procedures, or representations that reveal important information about the problem and contribute to its solution
    - Defining relevant variables and conditions carefully, including units if appropriate;
    - Seeking patterns and relationships
    - Looking for hidden structure
    - Considering special cases or simpler analogs
    - Applying previously learned concepts to new problem situations, adapting and extending as necessary
    - Making preliminary deductions and conjectures, including predicting what a solution to a problem might involve or putting constraints on solutions
    - Deciding whether a statistical approach is appropriate
2. Implementing a strategy
    - Making purposeful use of procedures
    - Organizing the solution
    - Making logical deductions
    - Monitoring progress toward a solution
3. Seeking and using connections across different mathematical domains, different contexts, and different representations
4. Reflecting on a solution to a problem
    - Interpreting a solution and how it answer the problem
    - Considering the reasonableness of a solution
    - Revisiting initial assumptions about the nature of the solution, including being mindful of special cases and extraneous solutions;
    - Justifying or validating a solution, including through proof or inferential reasoning;
    - Recognizing the scope of inference for a statistical solution
    - Reconciling different approaches to solving the problem
    - Refining arguments so that they can be effectively communicated
    - Generalizing a solution to a broader class of problems and looking for connections with other problems

Standards for Mathematical Practice in *Common Core State Standards in Mathematics* (CCSSI, 2010, p. 6-8)

- Make sense of problems and persevere in solving them
- Reason abstractly and quantitatively
- Construct viable arguments and critique the reasoning of others
- Model with mathematics
- Use appropriate tools strategically
- Attend to precision
- Look for and make use of structure
- Look for and express regularity in repeated reasoning

Descriptors of Process Standards in NCTM's *Principles and Standards for School Mathematics* (NCTM, 2000, p. 402)

1. Analyzing a problem
   - Build new mathematical knowledge through problem solving
   - Apply and adapt a variety of appropriate strategies to solve problems
   - Monitor and reflect on the process of mathematical problem solving
2. Reasoning and Proof
   - Make and investigate mathematical conjectures
   - Evaluate mathematical arguments and proofs
   - Use various types of reasoning and methods of proof
3. Communication
   - Organize and consolidate one's mathematical thinking
   - Analyze and evaluate the mathematical thinking and strategies of others
   - Use the language of mathematics to express mathematical ideas precisely
4. Connections
   - Recognize and use connections among mathematical ideas
   - Seek to understand how mathematical ideas interconnect
   - Seek to build a coherent network of ideas
5. Representation
   - Use representations to organize, record, and communicate mathematical ideas
   - Select, apply, and translate among mathematical representations to solev problems
   - Use representation to model and interpret physical, social, and mathematical phenomenon

Process Standards in Texas Essential Knowledge and Skills (TEA, 2012)

- apply mathematics to problems arising in everyday life, society, and the workplace
- use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying the solution, and evaluating the problem-solving process and the reasonableness of the solution
- select appropriate tools such as real objects, manipulatives, algorithms, paper and pencil, and technology and techniques such as mental math, estimation, number sense, and generalization and abstraction to solve problems
- effectively communicate mathematical ideas, reasoning, and their implications using multiple representations such as symbols, diagrams, graphs, computer programs, and language
- use mathematical relationships to generate solutions and make connections and predictions
- analyze mathematical relationships to connect and communicate mathematical ideas
- display, explain, or justify mathematical ideas and arguments using precise mathematical language in written or oral communication

Examples of Mathematical Practices in *Mathematical Practices for All* (RAND Mathematics Study Panel, 2003)

- Justifying claims
- Using symbolic notation efficiently
- Defining terms precisely
- Making generalizations
- Modeling a situation to make it easier to understand and to solve problems related to it
- Using algebraic notation cleverly to simplify a complex set of relationships
- Using geometric representation to make a problem transparent

Productive Habits of Mind (Marzano, 1992)

- Being sensitive to feedback
- Seeking accuracy and precision
- Persisting even when answers and solutions are not apparent
- Viewing situations in unconventional ways
- Avoiding impulsivity

Descriptors of Mathematical Thinking (Hull, Balka, & Miles, 2012)

- Attending to (or focusing on) the learning or problem
- Seeking holistic understanding of the situation or content
- Making sense of the problem or situation
- Drawing upon previous learning or knowledge
- Recalling prior experiences with similar situations
- Formulating ideas and questions

Descriptors of Mathematical Reasoning (Hull, Balka, & Miles, 2012)

- Removing extraneous or irrelevant information
- Applying logic to solve problems
- Using valid arguments to justify approaches and solutions
- Solving nontraditional or unfamiliar problems

Cross-disciplinary General Habits of Mind (Costa, 2000)

- Persisting
- Managing impulsivity
- Listening with understanding and empathy
- Thinking flexibly
- Thinking about thinking (metacognition)
- Striving for accuracy
- Questioning and posing problems
- Applying past knowledge to new situations
- Thinking and communicating with clarity and precision
- Gathering data through all senses
- Creating, imagining, innovating
- Responding with wonderment and awe

- Taking responsible risks
- Finding humor
- Thinking interdependently
- Remaining open to continuous learning

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