

# Selecting Representations

To adapt a hip-hop cultural expression, we math teachers “gotta represent!” In our context, NCTM (2000, p. 67) explains: “The term representation refers both to process and product—in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself.” Representation is also the most recently added standard of NCTM (2000) for school mathematics, the focus of the 2001 NCTM Yearbook, and the 2006-07 NCTM Professional Development Focus of the Year.

Choike (2000, p. 557) elaborates, “Students should be taught the value of representing mathematics verbally in words, numerically in tables, visually in graphs, and algebraically in symbols....A teaching strategy that connects the various forms of multiple representations to describe mathematics is an effective strategy for reaching out to students with different learning styles....Multiple representation is also a problem-solving strategy...an effective means for gaining insight into, and understanding of, a problem situation.” This also connects to the Texas Essential Knowledge and Skills (TEKS), of course. For example, section 6.12 of the 6th grade TEKS states: The student is expected to “communicate mathematical ideas using language, efficient tools, appropriate units, and graphical, numerical, physical, or algebraic mathematical models; and evaluate the effectiveness of different representations to communicate ideas.”

For the purposes of this article we shall refer to algebraic representation as analytic representation, which is used more frequently in the literature cited here. Hughes-Hallett (1991, p. 121) relates that, “One of the guiding principles is the ‘Rule of Three,’ which says that wherever possible topics should be taught graphically and numerically, as well as analytically. The aim is to produce a course where the three points of view are balanced...” Some mathematics educators add a verbal representation to make a ‘Rule of Four’, and sometimes the labels for representations are interpreted differently. The example in Table 1 shows the terms we will use in this article.

Abbreviation in this article	Representation Category Name	Actual Representation												
N	Numerical (or Tabular)	<table border="1"> <tr> <td>x</td> <td>0</td> <td>1</td> <td>2</td> <td>3</td> <td>4</td> </tr> <tr> <td>y</td> <td>1</td> <td>3</td> <td>5</td> <td>7</td> <td>9</td> </tr> </table>	x	0	1	2	3	4	y	1	3	5	7	9
x	0	1	2	3	4									
y	1	3	5	7	9									
G	Graphical													
A	Analytic (or Algebraic)	$f(x) = 2x + 1$												
V	Verbal	“output is one more than double the input”												

Table 1: Multiple Representations of a Linear Function

## Teaching with Representation: One versus Many

In the 1990’s, the second author implemented a pilot quasi-experimental design with seventy Russian high school precalculus students on solving inverse trigonometric identities such as  $\arctan(\frac{1}{2}) + \arctan(\frac{1}{3}) = \frac{\pi}{4}$  using multiple representations (Tchoshanov, 1997), as in Figure 1. The analytic/algebraic proof (where L and R denote the “left-hand side” and “right-hand side”, respectively) is the more familiar representation to most teachers and students. It is interesting to consider the geometric representation, which can be constructed on a 5x5 geoboard. The tangent of the angle with one marking (in the smaller shaded right triangle with  $1-2-\sqrt{5}$  sides) is  $\frac{1}{2}$ . The tangent of the angle with two markings (in the larger shaded right triangle with  $1-3-\sqrt{10}$  sides) is  $\frac{1}{3}$ . Those adjacent marked angles combine to form a larger angle in an isosceles right triangle so this new angle must be  $45^\circ$  or  $\frac{\pi}{4}$  radians.

The experiment consisted of two studies. The first study was focused on the effect of single and

combined representational modes on students' understanding and consisted of 3 groups. The first group of students was taught with a traditional analytic (algebraic) approach to trigonometric problem solving and proof. The second group was taught with a concrete/graphical approach using enactive (i.e., geoboard as manipulative aid) and graphical representations. The third group was taught with a combination of analytic and graphical means using translations among different representational modes.

As reported in Pape & Tchoshanov (2001), the combination group scored 26% higher than the graphical group and 43% higher than the analytic group. This experiment also showed that students in the "pure" (analytic or graphical) groups "stuck" to one particular mode of representation; they were reluctant to use different representations. For instance, students in the pure-graphical group tried to avoid any analytic solutions: they were "comfortable" if and only if they could use graphical techniques. Therefore, we realized that any intensive use of only one particular mode of representation does not improve students' conceptual understanding. Students in the combination group were much more flexible "switching" from one mode of representation to another in search of better understanding of mathematical concept. This observation supports findings from similar studies, such as Lesh, Post,

and Behr (1987, p. 38) who state that "good problem solvers tend to be sufficiently flexible in their use of a variety of relevant representational systems that they instinctively switch to the most convenient representation... at any given point in the solution process." Wilson (1994) found that being able to translate between multiple representations was deeply related to conceptual understanding.

NCTM (2000, p. 69) notes that "Different representations often illuminate different aspects of a complex concept or relationship," giving the example of various representations of fractions, including: sectors of a circle, fraction bars, points on a number line, and ratios of discrete elements of a set. Readers may wish to take a moment and think about what aspects of the fraction concept each of these representations emphasizes and what aspects are ignored. For example, fraction bars "convey the part-whole interpretation of fractions" but not "other interpretations of fraction, such as ratio, indicated division, or fraction as number."(ibid).

Multiple representations can be insightful even for the counting numbers. For example, Zazkis and Gadowsky (2001) ask which are perfect squares or cubes in the following set of numbers:

$$36^{200}, 36^{300}, 36^{400}, 36^{500}, 36^{600}, 36^{700}$$

They also ask for which of the following numbers can

#### Prove the Identity

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$$

#### Analytic/ Algebraic Proof

$$1. \tan(L) = \tan(R)$$

$$2. \tan(L) = \tan\left(\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)\right)$$

$$= \frac{\tan\left(\arctan\left(\frac{1}{2}\right)\right) + \tan\left(\arctan\left(\frac{1}{3}\right)\right)}{1 - \tan\left(\arctan\left(\frac{1}{2}\right)\right) \cdot \tan\left(\arctan\left(\frac{1}{3}\right)\right)}$$

$$= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}}$$

$$3. \tan(R) = \tan\left(\frac{\pi}{4}\right) = 1$$

Q.E.D.

#### Visual/ Geometric Proof

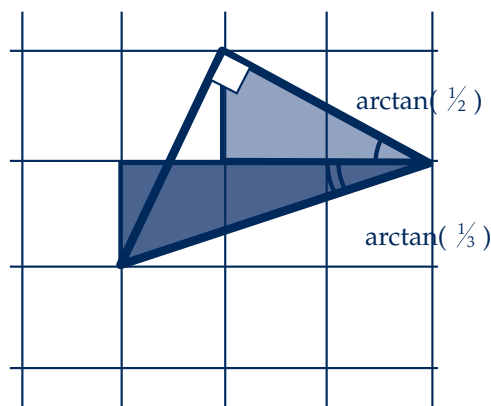


Figure 1: Trigonometric Identity Using Multiple Representations

one assess divisibility by 13 without calculation:

$$13^{50}, \quad 50^{13}, \quad 1000000000, \quad 123456 \times 13 + 3, \\ 39 + 654321 \times 13, \quad 36 \times 7654 + 3 \times 4567, \quad 24 \times 35 \times 56, \\ 24 \times 135 \times 56$$

(See representative solutions to these questions at the end of the article.) Teachers of all grade levels can create a surprisingly large variety of representations for even a single number such as 1.

## Sequence of Representations within a Collection

With evidence suggesting that a combination of representations is better, the next natural question for us was whether the sequence of representations within that combination may be significant. So the second study (later in the same semester) with the same classes of high school pre-calculus students was aimed at the effect that representational sequence has on students' understanding with respect to techniques of solving inverse trigonometric identities. We considered three different representational sequences, each placing the analytic representation in a different position (with numeric always preceding graphical).

The mean classroom test scores (out of 100 possible points) of the groups given the N-G-A, N-A-G and A-N-G sequences were 76, 85, and 91, respectively. These results appear to go against a dominant view among educators (especially those influenced by Piaget or Bruner) that mathematical activities should always be structured from concrete to abstract (i.e., the N-G-A sequence) in order to develop students' understanding of mathematical concepts and ultimately to improve students' performance. The learning model of Bruner (1966) is based upon three levels of engagement with representations: enactive (e.g., manipulating concrete materials), iconic (e.g., pictures and graphs), and then symbolic (e.g., analytic and algebraic).

However, Krutetskii (1976) shows that the differences in mathematical performance depend on mostly abstractness-oriented characteristics of the mathematical cast of mind. Vasilii Davydov (1990) first examined the effectiveness of the method of going from analytic to concrete by teaching algebra

concepts to typical Russian elementary school students in the early 1970's. Studies on Davydov's method have found that "the Russian students (from Davydov's program) have a profound grasp of mathematical structure, confidence, and the ability to extend their knowledge well beyond the levels at which they had been instructed" (Zeigenhagen, 2000). More discussion and interpretation on sequencing issues appears in Lesser and Tchoshanov (2005) and Van Patten, Chao & Reigeluth (1986).

Another reason the N-G-A sequence espoused by so many mathematics educators may not always be the best could be due to the particular mathematical content at hand. For example, consider Simpson's Paradox, which says that a comparison between two groups can be reversed when data is aggregated. Understanding this possibility is important for quantitative literacy, is listed as essential for citizenship (NCED 2001), and can be explored with only fraction arithmetic (see Table 2). As Lesser (2001, p. 131) notes: "It is routine to verify that within each department, women are hired at a higher rate than men, since  $30 \div 80 = 0.375 > 0.25 = 5 \div 20$  and  $15 \div 20 = 0.75 > 0.625 = 50 \div 80$ , yet are hired at a lower rate than men for the overall situation:  $(30 + 15) \div 100 = 0.45 < 0.55 = (5 + 50) \div 100$ ."

	Social sciences		Physical sciences	
	males	females	males	females
Hired	5	30	50	15
Not hired	15	50	30	5
Total	20	80	80	20

Table 2: Hiring Data by Gender and Department

Curriculum virtually always begins (and often also ends) with the numerical representation, which "is undeniably effective in demonstrating that Simpson's paradox can happen but limited in offering insight into how it can happen." (ibid). Of the ten or so representations in Lesser (2001), the most commonly used representation by statisticians is a graphical representation that can be called the trapezoidal representation (Tan 1986; Lesser 2004, 2005b), which seems to go against the end-goal of an abstract representation. As the y-intercepts of Figure 2 show, the female hiring rates for each department are higher

than the respective male hiring rates, but because of different gender mixtures for each department, the overall female hiring rate is nevertheless lower than the overall male hiring rate. Further discussion of this (and other representations) are in Lesser (2001, 2005b). Finally, it should be noted that several attendees at Lesser (2005a, 2005c) noted “blurring” between the three categories, such as finding both graphical and analytic features in the trapezoidal, circle graph, vector geometry and probability representations of Simpson’s Paradox from Lesser (2001), suggesting a modified “continuum” model of representation categorization.

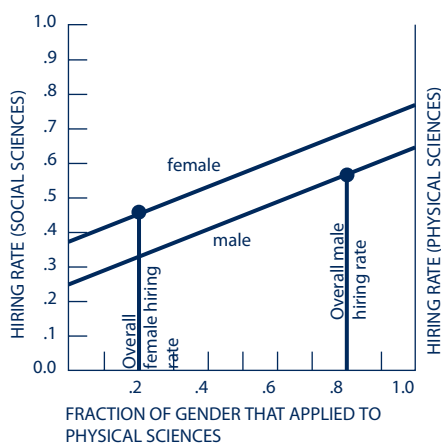


Figure 2: Trapezoidal Representation Adapted from Tan (1986)

## Teacher Experiences and Reflections

For some especially rich mathematical phenomena, the number of distinct representations may be too large to expect a teacher to have time to use all of them. Therefore, it is necessary to learn which representations might be more effective than others, as Lesser (2001) explored in pilot studies with pre-service secondary teachers (at a public research university and at a public comprehensive university) on exploring an abundance of representations of Simpson’s Paradox. It can be helpful to reflect upon these criteria (National Research Council (NRC) 2001, pp. 99-101) when selecting representation:

- transparency (“how easily can the idea be seen through the representation?”),
- efficiency (“does the representation support efficient communication and use?”),

- generality (“does the representation apply to broad classes of objects?”),
- clarity (“is the representation unambiguous and easy to use?”), and
- precision (“how close is the representation to the exact value?”).

To find out what real-life classroom factors might also influence teacher choice of representations or representational sequence, we surveyed mathematics teachers at a public middle school in El Paso County in spring 2005. Here are the factors that teachers said might influence their choice (in parentheses is the number of teachers that chose that factor from our list; we also gave them an option of “other”, but no one used it):

- learning style of students (7)
- teaching/ presentation style of teacher (7)
- particular math content involved (6)
- time constraints (6)
- learning goals (4)
- alignment with standardized tests or other assessment (3)

The differences between the NRC criteria and the classroom teachers’ criteria lead the authors to consider how all the criteria may be used most effectively. In the future, it would be worthwhile to examine further how the selection of particular representations and representational sequences may depend on each of these various factors.

## Professional Development

Under a 2005-06 Texas Education Agency grant, the authors delivered professional development workshops to help middle school teachers’ increase their students’ achievement in mathematics. Using item analysis of TAKS test data, the workshops engaged teachers in analyzing student error patterns and adapting pedagogy. As presented at the 2006 Conference for the Advancement of Mathematics Teaching and described in Lesser and Tchoshanov (in press),

Each session was launched by teacher reflections upon low-performing items from one of the 6 middle school TAKS objectives.

Teacher exploration of items went beyond teaching-to-the-test to unpacking big conceptual ideas and strategies (e.g., multiple representations) to help improve achievement on a much larger collection of items, and situate this understanding in a larger set of curriculum objectives and in the K-12 continuum.

While the average increase (from 2005 to 2006) in percentage points of our workshop’s teachers’ students passing the mathematics TAKS was significant (about 10 points), it is impossible to say how much of this was due to the particular strategy of multiple representations from our training.

### One Example

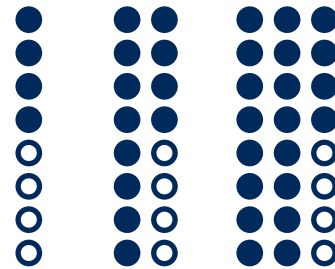
An example of a problem discussed in the workshops was this released item (#12) from the Spring 2004 7th grade mathematics TAKS test: “Generate a sequence that follows the rule  $8n-4$ , where  $n$  represents the position of a term in the sequence”. The choices were:

- (F) 16, 12, 8, 4, 0, .....
- (G) 8, 16, 24, 32, 40,.....
- (H) 4, 16, 64, 216, 1024,....
- (J) 4, 12, 20, 28, 36,.....

The inservice teachers discussed possible reasons why 7th graders’ success on this item was only 29%, barely better than what would be expected by random guessing. A plurality of students (42%) chose the incorrect answer (F). Trying this problem out with their students, teachers observed that students had difficulty “looking for the pattern from position to value of term” (as one teacher put it in her written reflections). The teachers agreed that students able to think of  $8n-4$  using multiple representations would be better equipped to solve the problem, and they generated many representations of this problem in our workshop discussion. For example, by representing  $8n-4$  as a line, the slope of 8 would suggest an incremental gain of 8 units when moving from one term to the next of a sequence (this observation alone already eliminates choices F and H). Teachers found they could also represent  $8n-4$  by forming an 8-by- $n$  rectangular array of stones and then remove 4 stones. Using ● for a stone and ○

for a removed stone, here’s what the pattern looks like for  $n = 1, 2, 3$ :

One teacher discovered that the strategy of actively



manipulating a given representation could also aid solving this problem. He noted that adding “4” to  $8n-4$  should result in a sequence of multiples of 8 and sure enough, only choice (J) yields multiples of 8 when 4 is added to all sequence values.

Representations will likely continue to be a rich focus for teachers and researchers seeking to understand and tap the power of representations to enhance student learning. Teachers whose students are struggling with algebraic representations may want to explore some of the resources that are available to teach mathematical ideas without using the algebraic representation. This can start as simple as illustrating the distributive property using a rectangle divided into two “subrectangles,” and can progress through a wide variety of further examples, such as those catalogued in Nelsen (1993, 2000). Statistical literacy textbooks such as Utts (2005) introduce basic concepts and even computations (e.g., for standard deviation) with their algebraic formulas deliberately delayed until the end of each chapter, after students gain intuition from other representations first. But even as educators explore what representations or what sequence of representations works best in each situation, it is helpful to keep in mind that the richest conceptual understanding and problem solving strength comes when students are able to translate readily from any representation to any other representation.

### Solutions:

Though too big for our calculators,  $36^{300}$  can be seen to be a square and a cube because the number can be represented as  $(36^{150})^2$  and  $(36^{100})^3$ , respectively.

We see that  $50^{13}$  is not a multiple of 13 because its unique prime factorization is a product of powers of only the primes 2 and 5:  $(2 \times 5^2)^{13}$ . From the division algorithm, we see that the representation  $123456 \times 13 + 3$  has a nonzero remainder (3) when divided by 13. From the distributive property, we see that  $39 + 654321 \times 13 = 13(3 + 654321)$ , a representation that indicates a multiple of 13.

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