THE EFFECT OF REPRESENTATION AND REPRESENTATIONAL SEQUENCE ON STUDENTS' UNDERSTANDING

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This study investigates the effect of representational sequence on students' understanding of mathematical concepts. Pilot studies were conducted with 129 high school students on solving inverse trigonometric identities and with 10 pre-service secondary teachers on representing Simpson's Paradox. Structured activities with a variety of representations and representational sequences were used to examine the impact on students' learning. This study also includes outcomes of surveys of 8 middle school teachers on different aspects of using representations in mathematics classroom. Our ongoing work finds this impact significant and claims that particular representational sequences need to be sensitive to specific content, learning outcomes, student prior knowledge and learning style.

Theoretical Framework

The body of existing research on the role of representation in improving students' mathematical understanding is convincing enough even for educational skeptics. That is why NCTM (2000) included the principle of representation among the five most important process standards of school mathematics. Studying an effect that representations have on students' understanding is critical for effectiveness of teaching mathematics. "We teach mathematics most effectively when we understand the effects on students' learning of external representations and structured mathematical activities" (Goldin & Shteingold 2001, p. 19).

Following Pape & Tchoshanov (2001), we use the term representation to refer to both the external and internal manifestations of mathematical concepts. Within the domain of mathematics, representations may be thought of as internal -- abstractions of mathematical ideas or cognitive schemata that are developed by a learner through experience. On the other hand, representations such as concrete, enactive models (e.g., manipulatives), visual/iconic (pictures, drawings, graphs, etc.), and symbolic/abstract (such as algebraic equations, formulas) are external manifestations of mathematical concepts that "act as stimuli on the senses" and help us understand these concepts (Janvier, Girardon, & Morand 1993, p. 81). Finally, representation also refers to the act of externalizing an internal, mental abstraction. In this paper we are mostly going to talk about the external representations.

A single type of external representation (e.g., visual) by itself doesn't ensure student learning and performance. Numerous studies (Brown & Presmeg 1993, Sylianou & Pitta-Pantazzi 2002) show that visualization is not always associated with mathematical accomplishment. Issues such as content, combination, and sequence of representations play a significant role in developing students' understanding. Lesh, Post, and Behr (1987) argue that not only are distinct types of external representation systems important in their own rights, "but translations among them, and transformations within them, also are important" (ibid., p. 34).

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Teaching with Representation: One Versus Many

In the 1990's, the second author implemented a pilot quasi experimental design with Russian high school pre calculus students (N=70) on solving inverse trigonometric identities such as arctan(1/2) + arctan(1/3) = pi/4 using multiple representations (Tchoshanov, 1997). The experiment consisted of two studies. The first study was focused on the effect of single and combined representational modes on students' understanding and consisted of 3 comparison groups. The first comparison group of students ("pure-analytic", n_1=23) was taught with a traditional analytic (algebraic) approach to trigonometric problem solving and proof. The second comparison group ("pure-visual", n_2=21) was taught with a visual (geometric) approach using enactive (i.e., geoboard as manipulative aid) and iconic (pictorial) representations. The third, experimental group ("representational", n_3=26) was taught with a combination of analytic and visual means using translations among different representational modes.

As reported elsewhere (Pape & Tchoshanov, 2001), the representational group scored 26% higher than the visual and 43% higher than the analytic groups. This experiment also showed that students in the "pure" (analytic and/or visual) groups "stuck" to one particular mode of representation; they were reluctant to use different representations. For instance, students in the pure-visual group tried to avoid any analytic solutions: they were "comfortable" if and only if they could use visual (geometric) techniques. Therefore, we realized that any intensive use of only one particular mode of representation does not improve students' conceptual understanding. Students in the representational group were much more flexible "switching" from one mode of representation to another in search of better understanding of mathematical concept. This observation supports findings from other similar studies; for example, one of the main conclusions from the study conducted by Lesh, Post, and Behr (1987) states that "good problem solvers tend to be sufficiently flexible in their use of a variety of relevant representational systems that they instinctively switch to the most convenient representation to emphasize at any given point in the solution process" (ibid., p. 38).

Sequence of Representations within a Collection

With evidence pointing in the direction that a combination of representations is the best, the next natural question was whether the sequence of representations within that combination may be significant. So the second study (later in the same semester) with the same classes of high school pre-calculus students was aimed at the effect that representational sequence has on students' understanding. Despite the tacitly accepted representational sequence "concrete-visual-abstract" where students first get involved into concrete "hands-on" experiences, then they draw the picture of the problem, and finally they provide formal (abstract, algebraic, or analytic) solution, we considered a variety of representational sequences. According to different types of representational sequence, there were 3 comparison groups (different from those participated in the first study with a total student sample of N=59):

1. The first comparison group (n_1=19) was called "abstract-last" group where activities were structured in a way that students first were engaged in concrete (C) and visual (V) modes of representation and only then to abstract-analytic (A) techniques of solving inverse trigonometric identities (CV-A & VC-A).
2. In the second - "abstract-middle" group (n_2=20) – activities were structured in a way that abstract (A) mode was introduced between concrete (C) and visual (V) modes of representation (CAV & VAC).

Activities in the third comparison group - “abstract-first” group (n_3=20) – were structured in the following representational sequence (ACV & AVC). According to this sequence, first students used trigonometric identities to simplify inverse trigonometric expressions and only then they were involved into activities using concrete and visual representations to illustrate and visually justify what they already proved analytically.

The mean classroom test scores in this study are the following: (1) Abstract-last group 76%; (2) Abstract-middle group - 85%; (3) Abstract–first group - 91%. At first glance, these results contradict the dominant view among educators that mathematical activities should be structured from concrete to abstract in order to develop students’ understanding of mathematical concepts and ultimately - to improve students’ performance. However, Krutetskii (1976) shows that the differences in mathematical performance depend on mostly abstractness-oriented characteristics of the mathematical cast of mind. Students in the Abstract-first group not only outperformed students from other two comparison groups but they were focusing on critical defining conditions (e.g., formalization, symbolization, generalization, curtailment, flexibility, and reversibility) of the problem and valued concrete and visual representation after they were exposed to analytic way of solving inverse trigonometric identities. Using the language of concept image and concept definition (Tall & Vinner, 1981), we may say that students in first two low-performing groups tended to have incomplete concept images without any connection to the defining conditions of the concepts, while students in the Abstract-first group attempted to use critical characteristics to form a concept definition. This result is in some way complementary to findings of Brown and Presmeg (1993), who claim that students with a greater relational understanding of mathematics tend to use more abstract forms of imagery, while students with less relational understanding tend to rely on concrete images. Overall, this study suggests that the students participating developed mathematical understanding that was enhanced not only by the combination (translation among and transformation within) of representations but also by the representational content and sequence.

Selection of Representations within a Collection

For some especially rich mathematical phenomena, the number of distinct representations may be too large to expect a teacher to have time to use all of them. Therefore, it is necessary to learn which representations might be more effective than others, and then form a sequence from those selected. Pilot studies were done by the first author with pre-service secondary teachers (7 at a public research university and 3 at a public comprehensive university) on exploring a sequence of 7 different representations of Simpson’s Paradox, following the examples in Lesser (2001). In this study, the focus was on determining which representations were more effective (and why) in helping students make sense out of a situation which did not seem very intuitive (at least in its first representation – a table of numbers): the results of a comparison were reversed upon aggregation of categories. Understanding this possibility is important for quantitative literacy. Students tended to want to stay with the most concrete and visual representations, but a C-V-A progression may not be expected to apply in the usual manner in the particular case of Simpson’s Paradox. We have followed up on this study and adapted the representations in a way that allows for students to be more active in their construction and interpretation (some of this was piloted in handouts during Lesser (2005) and more of it will be used in a future study with students). Also, we are beginning to explore the implications of the observation that some of the Lesser (2001) representations do not seem to neatly fit into only one of the three categories (C, V, A), suggesting a modified “continuum” model of representations.

Survey of In-service Teachers about Representation

Having observed results with pre-service teachers and pre-college students, the researchers felt that a missing part of the picture was how practicing in-service teachers themselves viewed representations. We conducted a survey in spring 2005 of the mathematics teachers (n = 8) at a public middle school in El Paso County with predominantly Hispanic population (93%) of students, the majority of which are economically disadvantaged (82%). Teachers participating in the interview have diverse teaching experience (3 teachers have 1-3 years of teaching experience, 3 have 4-6, and 2 have more than 6). The main purpose of the interview is to examine how teachers conceptualize various aspects of representation. They were asked to state in their own words what the NCTM (2000) representation standard meant and how it was part of their teaching. They were also asked to state in their own words the meaning of (and give examples of) concepts such as "concrete representation", "visual representation", and "algebraic representation". The examples they gave were analyzed to see if there were clear discrete separations between the categories or if the categories had some "blurring" into more of a continuum of representations. The survey then asked which sequence, if any, teachers thought was "best" (and why) or instead to discuss the manner in which one sequence might be best in some situations and another sequence in others. These answers were compared with the traditional curricular perception that the "best" sequence is "always" C-V-A, in the spirit of the simple-to-complex sequence supported by theories of Bruner (1996) and others. This sequence clearly follows Bruner's learning model based upon three levels of engagement with representations: enactive (e.g., manipulating concrete materials), iconic (e.g., pictures and graphs), and symbolic (e.g., numerals) (Bruner, 1966). Through early exploration of concrete materials, students are expected to move towards mathematical procedures that are analogous to symbolic procedures.

When the teachers were asked "When it comes to mathematics, what does 'representation' mean to you?" there were more than twice as many answers interpreting representation as a verb than as a noun. When asked to characterize their own learning style, many teachers said "traditional", but the rest of their answer made it clear this word did not have a common meaning. One teacher who described his style as analytical made the interesting follow-up comment: "But I am learning more in geometric representations. This is helping me become more versatile in my teaching."

Most of the teachers reported receiving very little beyond auditory lecture modes during their high school years, but finally experienced more visual, hands-on and kinesthetic styles when they took certain teacher preparation courses, especially in alternative certification classes. In their own teaching, teachers generally reported using multiple representations, believing that it helped reach more students, with one teacher noting the caveat "Would like to implement more except there is very little time."

Teachers were then asked to describe in their own words the meaning of (and give examples of) concrete, symbolic/abstract and visual representations. While the researchers had in mind that concrete was something like a table of numbers, teachers generally interpreted the term as physical objects or manipulatives, which is not unreasonable, but two teachers actually classified "formulas and algorithms" as concrete, possibly suggesting that these teachers were not used to classifying or articulating distinctions about multiple representations. The expert view of symbolic representation is formulas, equations, and algebraic notation. One teacher commented that symbolic "incorporates concrete".

When teachers were asked how they made their selection which representation (or sequence of representations) to use, the answers varied dramatically. Some teachers based the decision on their students (both by knowing their strengths beforehand, and by making adjustments if they still aren't getting it), some on a priori consideration of the content itself (e.g., "If I believe a concept is more visual, I would use a geometric teaching method first"), and some on the time available.

Of course, methods based on the essence of the concept would presume an accurate classification of a phenomenon's representation into a particular representation category. So the survey then asked if there are times when a representation might have strong aspects of more than one of the categories (visual, concrete, symbolic)? Again, student answers betrayed a lack of sophistication in representation classifications, but one student made a comment that revealed a provocative assumption: "There is research that attempts to prove that different representations confuses student[s], but I do not agree, I believe in using a combined approach to teaching Mathematics."

When asked what sequence (of the 6 permutations of the 3 categories) of representations the teachers thought was most effective, 50% chose the "traditional" answer of concrete, visual, abstract/symbolic, while the other half chose various different answers. One student justified her C-V-A choice with "because that's probably how are [sic] brain matures or develops." Once again, teachers were revealing some interesting beliefs about representations that we were not expecting. Another teacher who gave the C-V-A answer gave an explanation that conjured a progression of crystallizing thought: "I think concrete should come first, because it is the most clear picture. It is like connecting their learning to prior knowledge. Then I would put visual next because that is the next most clear concept. Then symbolic will be last because that requires more higher-order thinking skills."

The study suggests that the C-V-A choice is dominant in teachers' perceptions of effective representational sequences. It seems mostly due to their belief that "concrete should come first". This perception might be influenced by works of Bruner, Piaget, and others emphasizing method of ascending from concrete to abstract.

An informal survey of attendees (n = 6) at Lesser (2005) revealed a lack of consensus on the sequence with claims ranging from "sequencing should occur from simplest representation to more complex; start with concrete visual analytic" to "the sequencing is best done by knowing the audience that will be using the data." Also, several attendees at Lesser (2005) noted "blurring" between categories, such as finding visual and abstract features in the trapezoidal, circle graph, vector geometry and probability representations of Simpson's Paradox from Lesser (2001).

An alternative view on representational sequence is presented by Vygotskian fellow Vasilii Davydov. Davydov (1990) first examined the effectiveness of the method of ascending from general to concrete by teaching algebra concepts to elementary school students in the early 1970's in Russia. Studies on Davydov's method have found that "the Russian students (from Davydov's program) have a profound grasp of mathematical structure, confidence, and the ability to extend their knowledge well beyond the levels at which they had been instructed"(Zeigenhagen, 2000). Needless to say, that these students are from regular Russian schools without any selection by criterion of giftedness and a large percentage come from lower socioeconomic environments. Following Davydov's main idea, we argue that in students' conceptual development the representational sequence could be oriented toward ascending from whole to part, from abstract to concrete, from general concept to specific skill. It is not an imperative: the representational sequence depends on different factors such as content, learning objectives, student's prior experience and learning style.

In order to find out teachers' perception of this important issue, we asked them to identify factors that might influence their choice of a specific sequence of representations. Here are the factors that teachers said might influence their choice of a particular sequence (in parentheses is the number of teachers choosing that factor from our list; we also gave them an "other" option, but no one used it): learning style of students (7), teaching/ presentation style of teacher (7), particular math content involved (6), time constraints (6), learning goals (4), and alignment with standardized tests or other assessment (3). In this age of huge focus on high-stakes assessment, it is interesting that that factor was mentioned the least often.

Future Directions

We are planning a case study in which we explore the role of multiple contexts for the same mathematical structure. For the same Simpson's Paradox numbers in the 2x2x2 table of Lesser (2001), the variables of gender, department, and hiring decision can be replaced by other variables. For example, this distance/rate/time problem: "In January, Car A traveled at 37.5 mph for 80 hours of driving while Car B traveled at 25 mph for 20 hours of driving. In February, Car A traveled at 75 mph for 20 hours of driving while Car B traveled at 62.5 mph for 80 hours of driving. Which car had a lower rate of speed for both months combined?" Or a mixture problem: "Ann's box of mixed nuts consists of 80% walnuts priced at 37.5 cents/ounce and 20% cashews priced at 75 cents/ounce. Billy's box of mixed nuts consists of 20% walnuts priced at 25 cents/ounce and 80% cashews priced at 62.5 cents/ounce. Which box of mixed nuts is more expensive?" Or this example from the sports arena: "Sally, a guard on the women's basketball team, made 37.5% of her 80 attempts that were beyond the arc (i.e., 'three-pointers') and 75% of her 20 attempts that were close to the basket (i.e., 'in the paint'). Julie, a forward on the women's basketball team, made 25% of her 20 attempts from beyond the arc and 62.5% of her 80 attempts in the paint. Who had the higher shooting percentage overall?"

This future case study will also include several mathematical scenarios besides Simpson's Paradox for which each scenario can be represented with multiple representations (and contexts). Another example is a table of ordered pairs which would, say, all lie on a line when graphed and fit a $y=mx+b$ algebraic model. Even something as simple as a single number (e.g., one) can be represented with a picture, a word, a fraction, or expressions involving a zero exponent, a trigonometry function, etc. By exploring a variety of mathematical scenarios, we can gain insight into how the traditional C-V-A sequence might behave differently with different mathematical content, or how another sequence might be preferable for particular types or structures of content. In our future studies we also would like to explore the issue of representational "category blurring" or continuity (e.g., visualization as a continuum between concrete and abstract representations) and its impact on students' learning.

Outcomes of this study have direct implications for teacher preparation. Improvement of pre-service teachers' content knowledge depends on learning how to use representations and representational sequences effectively in the mathematics classroom.

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