Clssification of twisting of knots with less than eight crossings

ABSTRACT

In this paper, we study twisting of knots with less than seven crossings. We prove that all these knots are twisted except the seven crossing knot 7_5; depicted in Figure 3. Furthermore, we show that 7_5 is the smallest non-twisted prime knot.

1. Introduction

Throughout this paper, we work in the smooth category. All orientable manifolds will be assumed to be oriented unless otherwise stated. A knot is a smooth embedding of S^1 into the 3-sphere $S^3 \cong \mathbb{R}^3 \cup \{\pm\infty\}$. All knots are oriented.

Let K be a knot in the 3-sphere S^3, and D^2 a disk intersecting K in its interior. Let n be an integer. A $(-\frac{1}{n})$-Dehn surgery along $C = \partial D^2$ changes K into a new knot K_n in S^3. Let $\omega = \mathrm{lk}(\partial D^2, L)$. We say that K_n is obtained from K by (n,ω)-*twisting* (or simply *twisting*). Then we write $K \overset{(n,\omega)}{\to} K_n$, or $K \overset{(n,\omega)}{\to} K(n,\omega)$. We say that K_n is an (n,ω)-twisted (or simply twisted) knot provided that K is the unknot (Figure 1).

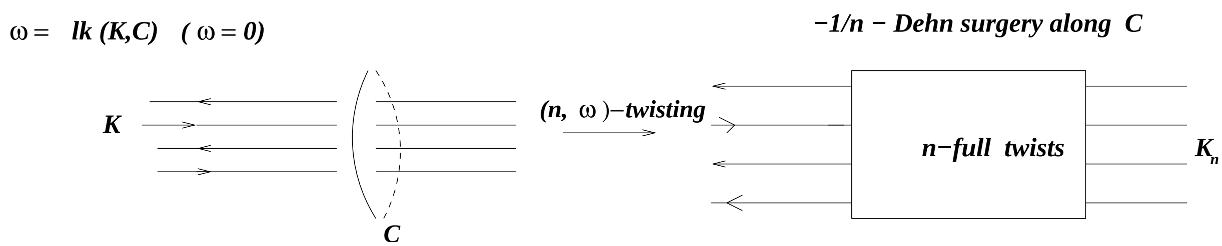

Figure 1:

An easy example is depicted in Figure 2, where we show that the right-handed trefoil $T(2,3)$ is obtained from the unknot $T(2,1)$ by a $(+1,2)$-twisting (In this case $n = +1$ and $\omega = +2$). Less obvious examples are given in Figure 7.

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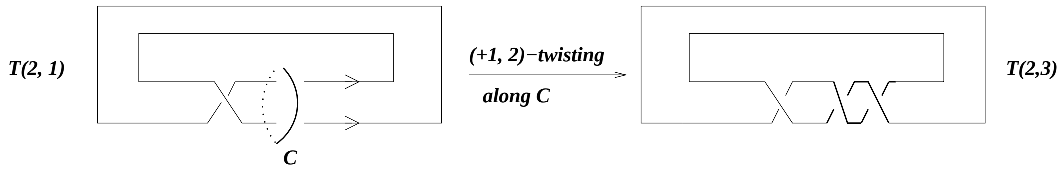

Figure 2:

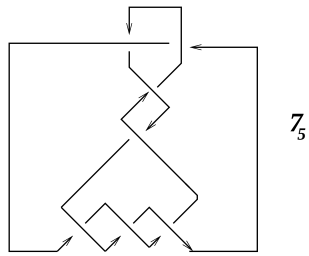

Figure 3:

Active research in twisting of knots started around 1990. One pioneer was the first author's Ph.D thesis advisor Y. Mathieu who asked the following questions in [13]:

Question 1.1. *Are all knots twisted ? If not, what is the minimal number of twisting disks ?*

To answer these questions, Y. Ohyama [17] showed that any knot can be untied by (at most) two disks, in one hand. In the other hand, K. Miyazaki and A. Yasuhara [15] were the first to give an infinite family of knots that are non-twisted. Furtheremore, they showed that the granny knot i.e. the product of two right-handed trefoil knots is the smallest non-twisted knot.

In his Ph.D. thesis [1] , the first author showed that the $(5, 8)$-torus knot is the smallest non-twisted torus knot. This was followed by his joint work with A. Yasuhara [4], in which an infinite family of non-twisted torus knots was given; using some techniques deriving from old gauge theory.

Hayashi-Motegi [9], and M. Teragaito [19] found independently examples of composite twisted knots. In addition, Hayashi-Motegi [9] and C. Goodman-Strauss [8] proved independently that, only single twisting (i.e. $| n |= 1$) can yield a composite knot.

This paper is the first of a series of joint papers with a group of undergraduate and graduate students, under the supervision of the first author; including A. Barsha, S. Inderias, T. Bui and A. Giragosian from the University of California at Riverside. We are interested in the following general question:

Question 1.2. Can we classify twisting of knots with less than ten crossings ?

With A. Barsha, we partially anwer this question by proving the following:

Theorem 1.1. *All knots with less than seven crossings are twisted, execpt 7_5. Furthermore, 7_5 is the smallest hyperbolic prime non-twisted knot.*

2. **Preliminaries**

There are several obstructions on twisting of knots deriving from congruence of knots and embedding of surfaces in 4-manifolds.

2.1. Congruence classes of Knots:

The notion of congruence classes of knots (due to R. H. Fox [6]) is an equivalence relation generated by certain twistings. A necessary condition for congruence is given by Nakanishi-Suzuki [16] in terms of Alexander polynomials.

Definition 2.1. [16] Let n, ω be non-negative integers. We say that a knot is ω-congruent to a knot L *modulo* n, ω and write $K \equiv^{\omega} L$ if there exist a sequence of knots $K = K_1, K_2, ..., K_m = L$ such that K_{i+1} can be obtained from K_i by an (n_i, ω)-twistings, where $n_i \equiv 0$ (mod. n)

Theorem 2.1. (Y. Nakanishi and S. Suzuki [16]) If $K \equiv^{\omega} L$ then

(1) $\Delta_K(t) \pm t^r \Delta_L(t)$ is a multiple of $(1-t)\sigma_n(t^{\omega})$ for some integer r; where $\sigma_n(t) = \dfrac{t^n - 1}{t-1}$.

(2) If n or ω is even, then $\Delta_K(t) \equiv \Delta_L(t)$ (mod. $2n$).

Example 2.1. Figure 8 shows that $7_5 \equiv^2 K$; where K is the unknot.

Example 2.2. If K_n is obtained from the unknot K by (n, ω)-twisting, then $K_n \equiv^{\omega} K$. In particular, Theorem 2.1 applies to twisted knots.

2.2. Embedding of surfaces in 4-manifolds:

In the following, $b_2^+(X)$ (resp. $b_2^-(X)$) is the rank of the positive (resp. negative) part of the intersection form of an oriented, compact 4-manifold X. Let $\sigma(X)$ denote the signature of X. Then a class $\xi \in H_2(X, \mathbb{Z})$ is said to be characteristic provided that $\xi.x \equiv x.x$ for any $x \in H_2(X, \mathbb{Z})$ where $\xi.x$ stands for the pairing of ξ and x, i.e. their Kronecker index and ξ^2 for the self-intersection of ξ in $H_2(X, \mathbb{Z})$.

The following theorem is originally due to O.Ya. Viro [21]. It is also obtained by letting $a = [d/2]$ in the inequality of [7, Remarks(a) on p-371] by P. Gilmer. Let $\sigma_d(K)$ denotes the Tristram's signature of K [20].

Theorem 2.2. Let X be an oriented, compact 4-manifold with $\partial X = S^3$, and K a knot in ∂X. Suppose K bounds a surface of genus g in X representing an element ξ in $H_2(X, \partial X)$.

(1) If ξ is divisible by an odd prime d, then: $\mid \dfrac{d^2-1}{2d^2}\xi^2 - \sigma(X) - \sigma_d(K) \mid \leq dim H_2(X; \mathbb{Z}_d) + 2g$.

(2) If ξ is divisible by 2, then: $\mid \dfrac{\xi^2}{2} - \sigma(X) - \sigma(K) \mid \leq dim H_2(X; \mathbb{Z}_2) + 2g$.

Theorem 2.3. (K. Kikuchi [11]) Let X be a closed, oriented simply connected 4-manifold with $b_2^+(X) \leq 3$ and $b_2^-(X) \leq 3$. Let ξ be a characteristic element of $H_2(X;\mathbb{Z})$. If ξ is represented by a 2-sphere, then

$$\xi \cdot \xi = \sigma(X).$$

2.3. The minimal genus problem:

The genus function G is defined on $H_2(X,\mathbb{Z})$ as follows: For $\alpha \in H_2(X,\mathbb{Z})$, consider

$$G(\alpha) = min\{genus(\Sigma) | \Sigma \subset X \quad represents \quad \alpha, i.e., [\Sigma] = \alpha\}$$

where Σ ranges over closed, connected, oriented surfaces smoothly embedded in the 4-manifold X. Note that $G(-\alpha) = G(\alpha)$ and $G(\alpha) \geq 0$ for all $\alpha \in H_2(X,\mathbb{Z})$ (see Gompf-Stipsicz [8]).

Theorem 2.4 (D. Ruberman) Let $\alpha = [S^2 \times pt.]$ and $\beta = [pt. \times S^2]$ be the standard generators of $H_2(S^2 \times S^2, \mathbb{Z})$ with $\alpha \cdot \alpha = \beta \cdot \beta = 0$ and $\alpha \cdot \beta = 1$. If $ab \neq 0$ then

$$G(a\alpha + b\beta) = (| a | -1)(| b | -1).$$

Obviously $G(a\alpha) = G(b\beta) = 0$.

3. Proof of Theorem 1.1

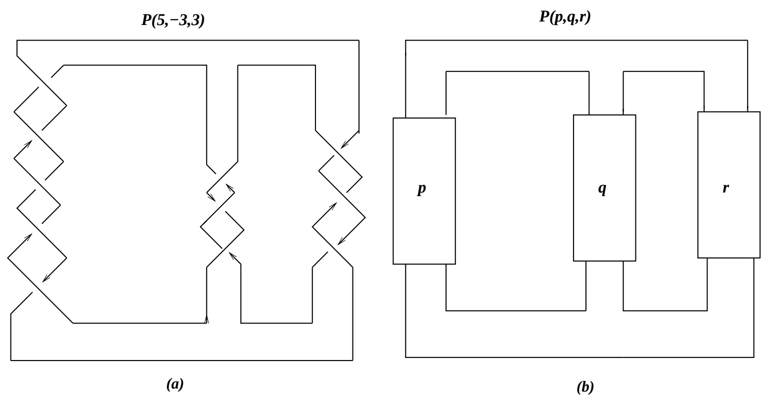

Figure 4:

To prove Theorem 1.1, we need to prove Proposition 3.1. and Proposition 3.2.

In the following, let $P(p,q,r)$ denote the 3-stranded pretzel knot with p,q and r half-twists in its strands as in Figure 4(b). An example is illustrated in Figure 4(a) with $(p,q,r) = (5,-3,3)$. It is well-known that $P(p,q,r)$ is a knot if and only if p,q and r are odd.

Proposition 3.1. Let k be a positive knot that is not the connected sums of *pretzel knots* $P(p,q,r)$ (p,q and r are odd). If k is an (n,ω)-twisted knot with $\omega \neq \pm 1$, then $n > 0$.

Proof. In [14], J.H. Przytycki and K. Taniyama showed that, except for connected sums of *pretzel knots* $P(p,q,r)$ (pqr is odd), a *positive knot* can be deformed into $T(2,5)$ by changing some positive crossings to be negative. Since $\sigma_d(T(2,5)) = -4$ for any prime integer d ([20, Lemma 3.5]), then by [4, Lemma 3.4] , we have $\sigma_d(k) \leq -4$.

Assume now that $n < 0$ and $\omega \neq \pm 1$, then k bounds a disk $(\Delta, \partial\Delta) \subset (\mid n \mid \mathbb{C}P^2 - intB^4, S^3)$ such that $[\Delta] = \omega(\gamma_1 + ... + \gamma_{|n|})$ in $H_2(\mid n \mid \mathbb{C}P^2 - intB^4, S^3; \mathbb{Z})$ and $\gamma_1, \gamma_2, ..., \gamma_{|n|}$ are the standard generators of $H_2(\mid n \mid \mathbb{C}P^2 - intB^4, S^3; \mathbb{Z})$ with the intersection number $\gamma_i \cdot \gamma_j = \delta_{ij}$; where $1 \leq i,j \leq n$ and δ_{ij} is the Kronecker's delta $\delta_{ij} = \begin{cases} 1 & if \quad i = j. \\ 0 & if \quad i \neq j. \end{cases}$

Case 1. If ω is odd, then let $d > 2$ denote the smallest prime divisor of ω. Gilmer-Viro's Theorem yields that $\mid\mid n \mid \omega^2 \frac{d^2-1}{2d^2} - \mid n \mid -\sigma_d(k) \mid \leq \mid n \mid$. Or equivalently, $\mid\mid n \mid (\frac{d^2-1}{2}(\frac{\omega}{d})^2 - 1) - \sigma_d(k) \mid \leq \mid n \mid$. This would contradict that $\sigma_d(k) \leq -4$.

Case 2. If ω is even, then Gilmer-Viro's Theorem yields that $\mid \frac{\mid n \mid \omega^2}{2} - \mid n \mid -\sigma(k) \mid \leq \mid n \mid$. Or equivalently, $\mid\mid n \mid (\frac{\omega^2}{2} - 1) - \sigma(k) \mid \leq \mid n \mid$. Henceforth, the only possibilities are: $\omega = 0$ and $\sigma(k) = 0$ or -2; or $\omega = \pm 2$ and $\sigma(k) = 0$. This would contradict that $\sigma(k) \leq -4$.

Proposition 3.2. Let k be an (n,ω)-twisted knot, where n and $\omega(\neq 0)$ are both even, then the following inequality holds:

$$(\mid \omega \mid -1)(\mid \frac{n\omega}{2} \mid -1) \leq g^*.$$

where g^* is the 4-ball genus of k.

Proof. Assume that k is an (n,ω)-twisted knot, where n and $\omega(\neq 0)$ are both even. Then k bounds a disk $(\Delta, \partial\Delta) \subset (S^2 \times S^2 - intB^4, S^3)$ such that $\partial\Delta = k$ and

$$[\Delta] = -\epsilon\omega\alpha + \frac{n\omega}{2}\beta \in H_2(S^2 \times S^2 - intB^4, S^3; \mathbb{Z})$$

with $\epsilon = sign(n)$ (See Lemma 3.2 in [4]. See also K. Miyazaki and A. Yasuhara [15], Fig. 4 on p-146 as well as T. Cochran and R. E. Gompf [5], Fig. 12 on p-506). Let $(S_{g^*}, \partial S_{g^*}) \subset (intB^4, \partial B^4 \cong S^3)$ be a compact, connected and oriented surface such that $\partial S_{g^*} = \bar{k}$, where $\bar{k} = -k^*$ is the dual knot of k i.e. the inverse of the mirror image of k [10]. Gluing Δ and S_{g^*} along their boundaries k yields a smooth closed genus g^* surface $\Sigma_{g^*} = \Delta \bigcup_k S_{g^*}$ embedded in $S^2 \times S^2$. By Ruberman's theorem we have $G((\pm\omega, \frac{n\omega}{2}) = (\mid \omega \mid -1)(\mid \frac{n\omega}{2} \mid -1)$ in $H_2(S^2 \times S^2 - intB^4, S^3; \mathbb{Z})$. Therefore, $(\mid \omega \mid -1)(\mid \frac{n\omega}{2} \mid -1) \leq g^*$.

As a corrolary of Proposition 3.2.

Corrolary 3.1. If k is an (n, ω)-twisted knot with $g^* = 1$, where n and ω are both even, then $\omega = 0$ or $(n, \omega) = (\pm 2, \pm 2)$.

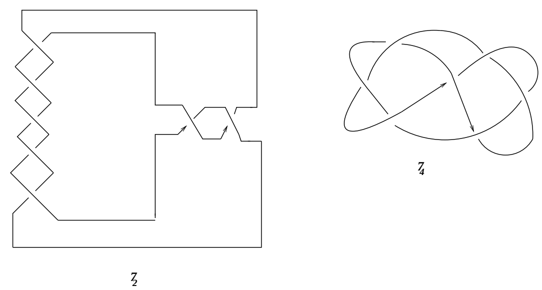

Figure 5:

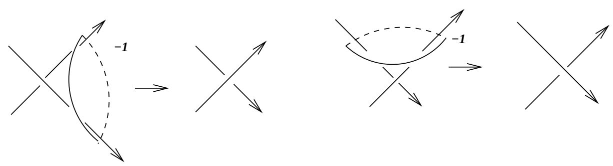

Figure 6:

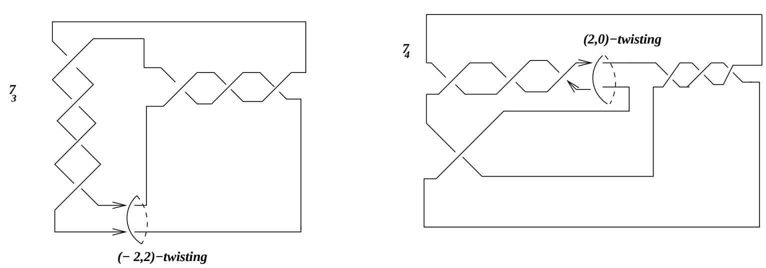

Figure 7:

Lemma 3.1. The only pretzel knots with less than eight crossings are:

$$7_2 = P(5, 1, 1) \quad \text{and} \quad 7_4 = P(3, 1, 3).$$

Proof. If $P(p, q, r)$ is a 3-stranded pretzel knot with less than eight crossings, then either $(p, q, r) = (5, 1, 1)$ or $(p, q, r) = (3, 1, 3)$. Figure 5 shows that $7_2 = P(5, 1, 1)$ and $7_4 = P(3, 1, 3)$ (The projection of 7_2 (resp. 7_4) is illustrated in knotinfo [2] (resp. in [18, D. Rolfsen])).

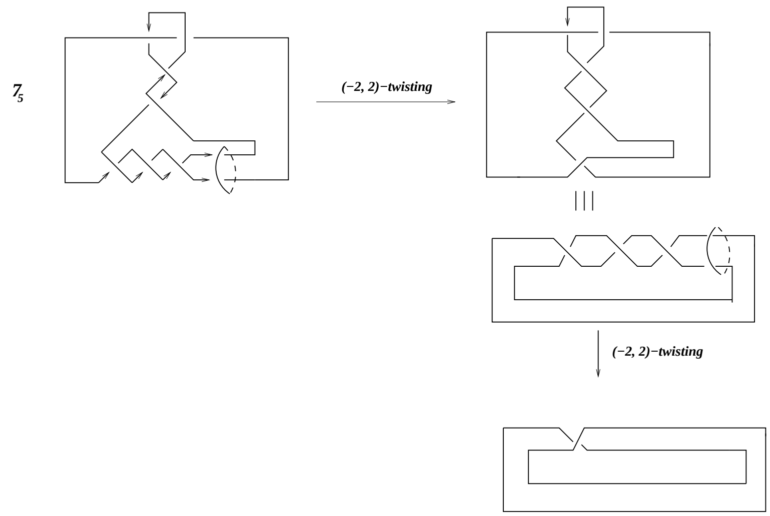

Figure 8:

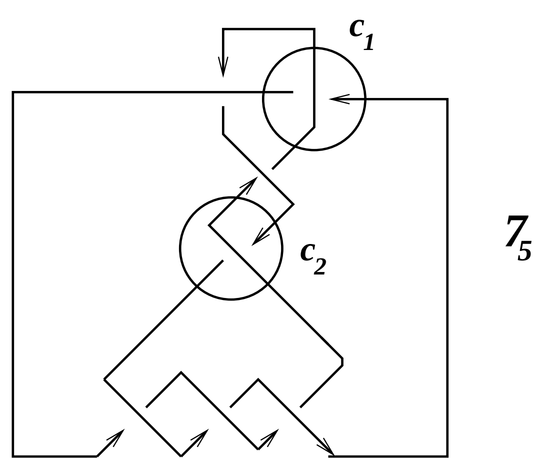

Figure 9:

Proof of Theorem 1.1. As depicted in Figure 6, we can easily check that any unknotting number one knot is (-1)-twisted. Note that all knots with less than seven crossings, except $7_1, 7_3, 7_4$ and 7_5, are unknotting number one knots [10, A. Kawauchi]; and therefore they are (-1)-twisted. In the other hand, it is easy to see that the $(2,7)$-torus knot 7_1 is $(+3,2)$-twisted, and Figure 7 shows that 7_3 is $(+2,2)$-twisted and 7_4 is $(-2,0)$-twisted. Therefore, it remains to prove that 7_5 is a non-twisted knot.

Assume, for a contradiction, that 7_5 is obtained by an (n,ω)-twisting from an unknot K along an unknot C; where n is the number of twistings and $\omega = lk(K,C)$. Note that 7_5 is a positive knot that is not the connected sum of pretzel knots (see Lemma 3.1). By virtue of Proposition 3.1, we have $n > 0$ or $\omega = \epsilon$ where $\epsilon = \pm 1$.

Case 1. If $\omega = \epsilon$, then this would contradict Suzuki-Nakanishi's theorem. Indeed, we would have

$$det(7_5) \equiv \pm t^r (mod. \quad (t^{\epsilon n} - 1)).$$

Letting $t = -1$ if n is even and $t = -1$ if n is odd would imply that $det(7_5) = \pm 1$; a contradiction.

Case 2. If $n > 0$, then 7_5 bounds a disk $(\Delta, \partial\Delta) \subset (n\overline{\mathbb{C}P^2} - intB^4, S^3)$ such that

$$[\Delta] = \omega(\bar{\gamma}_1 + ... + \bar{\gamma}_n) \in H_2(n\overline{\mathbb{C}P^2} - intB^4, S^3; \mathbb{Z}).$$

where $\bar{\gamma}_1, \bar{\gamma}_2, ..., \bar{\gamma}_n$ are the standard generators of $H_2(n\overline{\mathbb{C}P^2} - intB^4, S^3; \mathbb{Z})$ with the intersection number $\bar{\gamma}_i \cdot \bar{\gamma}_j = -\delta_{ij}$; where δ_{ij} is the Kronecker's delta.

Case 2.1. If ω is even, then Theorem 2.2 yields that $\mid -\dfrac{n\omega^2}{2} - (-n) - \sigma(7_5) \mid \leq n$. Or equivalently, $\mid n(\dfrac{\omega^2}{2} - 1) - 4 \mid \leq n$. Therefore, the only remaining possibilities to preclude are $\omega = \pm 2$ and $n \geq 2$.

If $\omega = \pm 2$ and $n \geq 2$ is even, then Corrolary 3.1 yields that $(n,\omega) = (2, \pm 2)$. Therefore, 7_5 bounds a properly embedded disk $D \subset S^2 \times S^2 - intB^4$ such that $[D] = \mp 2\alpha_1 + 2\beta_1 \in H_2(S^2 \times S^2 - intB^4, S^3, \mathbb{Z})$ and $\partial D = 7_5$. Figure 8 shows that $U \overset{(-2,2)}{\longrightarrow} T(-2,3) \overset{(-2,2)}{\longrightarrow} \bar{7}_5$, where $\bar{7}_5$ is the dual knot of 7_5. Then there exists a properly embedded disk $\Delta \subset S^2 \times S^2 \# S^2 \times S^2 - intB^4$ such that $\partial\Delta = \bar{7}_5$; and $[\Delta] = 2\alpha_2 + 2\beta_2 + 2\alpha_3 + 2\beta_3 \in H_2(S^2 \times S^2 \# S^2 \times S^2 - intB^4, S^3; \mathbb{Z})$. Therefore, the sphere

$$[S] = [D \cup \Delta] = \mp 2\alpha_1 + 2\beta_1 + 2\alpha_2 + 2\beta_2 + 2\alpha_3 + 2\beta_3 \in H_2(3S^2 \times S^2, \mathbb{Z}).$$

is a characteristic class, which would contradict Kikuchi's Theorem.

If $\omega = \pm 2$ and $n \geq 2$ is odd, then by Nakanishi-Suzuki's theorem, $\Delta_{7_5}(-1) \equiv \Delta_K(-1)$ (mod. $2n$). This implies that $17 \equiv 1$ (mod. $2n$); or equivalently, $n = \pm 1$; a contradiction.

Case 2.2. If ω is odd, then let $d > 2$ denote the smallest prime divisor of ω. Gilmer-Viro's Theorem yields that $\mid -n\omega^2 \dfrac{d^2-1}{2d^2} - (-n) - \sigma_d(7_5) \mid \leq n$. Figure 9 shows that the right-handed trefoil (resp. the $(2,5)$-torus) knot can be obtained from 7_5 by changing the positive crossing c_1 (resp. c_2) into negative one, then by Lemma 3.4 in [4], for any prime integer d

$$\begin{cases} \sigma_d(T(2,5)) - 2 \leq \sigma_d(7_5) & \leq \quad \sigma_d(T(2,5)). \\ \sigma_d(T(2,3)) - 2 \leq \sigma_d(7_5) & \leq \quad \sigma_d(T(2,3)). \end{cases}$$

Since $\sigma_d(T(2,3)) = -2$ and $\sigma_d(T(2,5)) = -4$ [3, Lemma 4.1]; then $\sigma_d(7_5) = -4$. This implies that the only remaining possibilities to preclude are $(n, \omega) = (1, 3\epsilon)$ or $(n, \omega) = (2, 3\epsilon)$ with $\epsilon = sign(\omega)$.

If $(n, \omega) = (1, 3\epsilon)$, then 7_5 bounds a disk $(\Delta, \partial\Delta) \subset (\overline{\mathbb{C}P^2} - intB^4, S^3)$ such that $[\Delta] = 3\epsilon\bar{\gamma}$ in $H_2(\overline{\mathbb{C}P^2} - intB^4, S^3; \mathbb{Z})$, where $\bar{\gamma}$ is the standard generator of $H_2(\overline{\mathbb{C}P^2} - intB^4, S^3; \mathbb{Z})$ with $\bar{\gamma} \cdot \bar{\gamma} = -1$. By an argument similar to that in **Case 2.1** above would contradict Kikuchi's theorem.

If $(n, \omega) = (2, 3\epsilon)$, then this would contradict Suzuki-Nakanishi's theorem. Indeed, we would have:

$$det(7_5) \equiv \pm t^r (mod.(1-t)(1+t^{3\epsilon})).$$

If we let $t = -1$, then we would have $det(7_5) = \pm 1$; a contradiction.

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