Classification of twisting of knots with less than eight crossings

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ABSTRACT

In this paper, we study twisting of knots with eight crossings. We show that 8_{15} is the smallest eight crossing non-twisted prime knot.

[1] [2] 1. Introduction

Throughout this paper, we work in the smooth category. All orientable manifolds will be assumed to be oriented unless otherwise stated. A knot is a smooth embedding of S^1 into the 3-sphere $S^3 \cong \mathbb{R}^3 \cup \{\pm\infty\}$. All knots are oriented.

Let K be a knot in the 3-sphere S^3, and D^2 a disk intersecting K in its interior. Let n be an integer. A $(-\frac{1}{n})$-Dehn surgery along $C = \partial D^2$ changes K into a new knot K_n in S^3. Let $\omega = \mathrm{lk}(\partial D^2, L)$. We say that K_n is obtained from K by (n, ω)-*twisting* (or simply *twisting*). Then we write $K \overset{(n,\omega)}{\to} K_n$, or $K \overset{(n,\omega)}{\to} K(n, \omega)$. We say that K_n is an (n, ω)-twisted (or simply twisted) knot provided that K is the unknot.

An easy example is depicted in Figure 2, where we show that the right-handed trefoil $T(2, 3)$ is obtained from the unknot $T(2, 1)$ by a $(+1, 2)$-twisting (In this case $n = +1$ and $\omega = +2$). Less obvious examples are given in Figure 7.

Active research in twisting of knots started around 1990. One pioneer was the first author's Ph.D thesis advisor Y. Mathieu who asked the following questions in [12]:

Question 1.1. *Are all knots twisted ? If not, what is the minimal number of twisting disks ?*

To answer these questions, Y. Ohyama [15] showed that any knot can be untied by (at most) two disks, in one hand. In the other hand, K. Miyazaki and A. Yasuhara [13] were the first to give an infinite family of knots that are non-twisted. Furtheremore, they showed that the granny knot i.e. the product of two right-handed trefoil knots is the smallest non-twisted knot.

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[1] *2000 Mathematics Subject Classification.* 57M25, 57M45

[2] *Key Words and Phrases.* Twisted Knots, congruence classes of knots, minimum genus function.

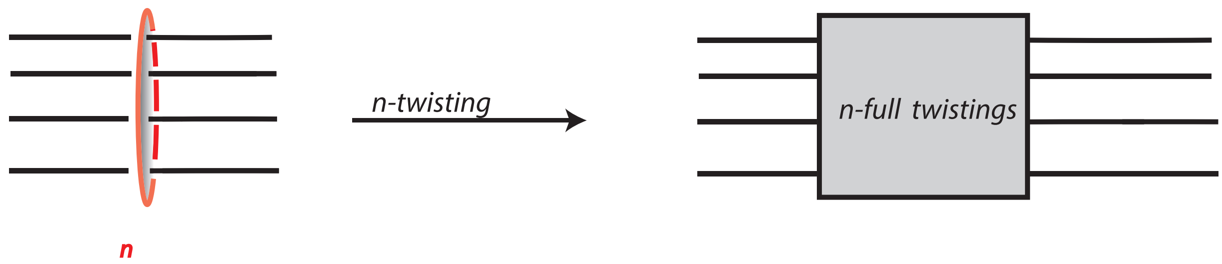

Figure 1:

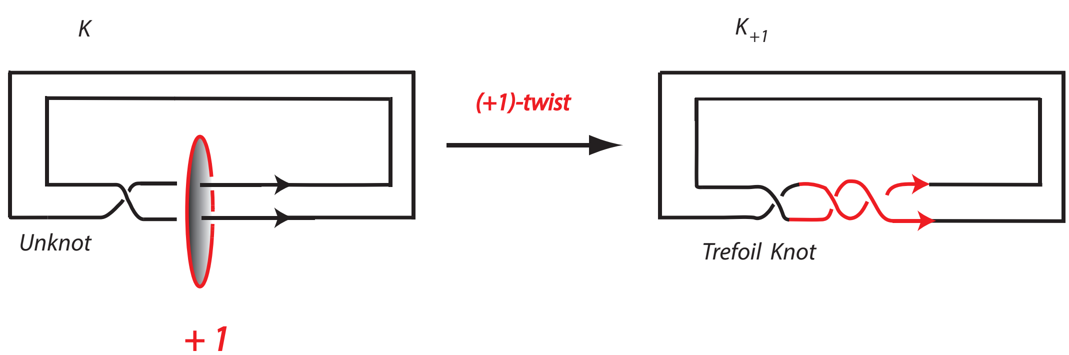

Figure 2:

In his Ph.D. thesis [1] , the first author showed that the $(5,8)$-torus knot is the smallest non-twisted torus knot. This was followed by his joint work with A. Yasuhara [5], in which an infinite family of non-twisted torus knots was given; using some techniques deriving from old gauge theory.

We are interested in the following general question:

Question 1.2. Can we classify twisting of knots with less than ten crossings ?

In [4], we proved that all knots with less than eight crossings are twisted except 7_5. In this paper, we partially answer this question by proving the following:

Theorem 1.1. *All knots prior to* 8_{16} *are twisted, except* 7_5 *and* 8_{15} *.*

2. **Preliminaries**

There are several obstructions on twisting of knots deriving from congruence of knots and embedding of surfaces in 4-manifolds.

2.1. Congruence classes of Knots:

The notion of congruence classes of knots (due to R. H. Fox [7]) is an equivalence relation generated by certain twistings. A necessary condition for congruence is given by Nakanishi-Suzuki [14] in terms of Alexander polynomials.

Definition 2.1. [14] Let n, ω be non-negative integers. We say that a knot is ω-congruent to a knot L *modulo* n, ω and write $K \equiv^{\omega} L$ if there exist a sequence of knots $K = K_1, K_2, ..., K_m = L$ such that K_{i+1} can be obtained from K_i by an (n_i, ω)-twistings, where $n_i \equiv 0$ (mod. n)

Theorem 2.1. (Y. Nakanishi and S. Suzuki [14]) If $K \equiv^{\omega} L$ then

(1) $\Delta_K(t) \equiv \pm t^r \Delta_L(t)$ is a multiple of $(1-t)\sigma_n(t^{\omega})$ for some integer r; where $\sigma_n(t) = \dfrac{t^n - 1}{t-1}$.

(2) If n or ω is even, then $\Delta_K(-1) \equiv \Delta_L(-1)$ (mod. $2n$).

Whar figure **Example 2.1.** Figure 8 shows that $8_{15} \equiv^2 K$; where K is the unknot.

Example 2.2. If K_n is obtained from the unknot K by (n, ω)-twisting, then $K_n \equiv^{\omega} K$. In particular, Theorem 2.1 applies to twisted knots.

2.2. Embedding of surfaces in 4-manifolds:

In the following, $b_2^+(X)$ (resp. $b_2^-(X)$) is the rank of the positive (resp. negative) part of the intersection form of an oriented, compact 4-manifold X. Let $\sigma(X)$ denote the signature of X. Then a class $\xi \in H_2(X, \mathbb{Z})$ is said to be characteristic provided that $\xi.x \equiv x.x$ for any $x \in H_2(X, \mathbb{Z})$ where $\xi.x$ stands for the pairing of ξ and x, i.e. their Kronecker index and ξ^2 for the self-intersection of ξ in $H_2(X, \mathbb{Z})$.

The following theorem is originally due to O.Ya. Viro [18]. It is also obtained by letting $a = [d/2]$ in the inequality of [8, Remarks(a) on p-371] by P. Gilmer. Let $\sigma_d(K)$ denotes the Tristram's signature of K [17].

Theorem 2.2. Let X be an oriented, compact 4-manifold with $\partial X = S^3$, and K a knot in ∂X. Suppose K bounds a surface of genus g in X representing an element ξ in $H_2(X, \partial X)$.

(1) If ξ is divisible by an odd prime d, then: $\mid \dfrac{d^2-1}{2d^2}\xi^2 - \sigma(X) - \sigma_d(K) \mid \leq dim H_2(X; \mathbb{Z}_d) + 2g$.

(2) If ξ is divisible by 2, then:

Theorem 2.3. (P. Ozsváth and Z. Szabó [16]) Let W be a smooth, oriented four-manifold with $b_2^+(W) = b_1(W) = 0$, and $\partial W = S^3$. If Σ is any smoothly embedded surface-with-boundary in W whose boundary lies on S^3, where it is embedded as the knot K, then we have the following inequality:

$$\tau(K) + \frac{\mid [\Sigma] \mid + [\Sigma].[\Sigma]}{2} \leq g(\Sigma).$$

Theorem 2.4 (K. Kikuchi [10]) Let X be a closed, oriented and smoothy 4-manifold such that:

(1) $H_1(X)$ has no 2-torsion;

(2) $b_2^{\pm}3$.

If $\xi \in H_2(X, \partial X)$ is a characteristic class, then we have:

$$\xi^2 = \sigma(X).$$

The genus function G is defined on $H_2(X, \mathbb{Z})$ as follows: For $\alpha \in H_2(X, \mathbb{Z})$, consider

$$G(\alpha) = min\{genus(\Sigma)|\Sigma \subset X \quad represents \quad \alpha, i.e., [\Sigma] = \alpha\}$$

Where Σ ranges over closed, connected, oriented surfaces smoothly embedded in the 4-manifold X. Note that $G(-\alpha) = G(\alpha)$ and $G(\alpha) \geq 0$ for all $\alpha \in H_2(X, \mathbb{Z})$ (An excellent reference is Gompf-Stipsicz [9]).

Theorem 2.5 (D. Ruberman) The minimum genus of a smooth surface representing the class $a\alpha + b\beta$ in $S^2 \times S^2$ is

$$G(a\alpha + b\beta) = (\mid a \mid -1)(\mid b \mid -1).$$

when a and b are not zero. If $a = 0$ or $b = 0$, then the class can be represented by a sphere. sphere.

Proof of Theorem 1.1.

Since 8_{15} is a positive knot, then there are two cases

Case 1 $n > 0$.

Case 1.1. If $\omega = 1$, then by Theorem 2.3,

$$\tau(8_{15}) + \frac{n \mid \omega \mid - n\omega^2}{2} \leq 0.$$

This would contradict that $\tau(8_{15}) = 2$.

Case 1.2. Assume now that ω is even, then by Gilmer Viro

$$\mid -\frac{n\omega^2}{2} - \sigma(8_{15}) + n \mid \leq n,$$

or equivalently,

$$| n(\frac{\omega^2}{2} - 1) - 4 | \leq n$$

This yields that $\omega = 2$ is the only posibility. By Theorem 2.1, $\triangle_{8_{15}}(-1) \equiv \triangle_U(-1)\,(mod\,2n)$, or equivalently, $33 \equiv 1\,(mod\,2n)$. Therefore, $n \in \{1, 2, 4, 8, 16\}$. The case $n = +1$ is excluded by Theorem 2.

Assume now that $n \in \{2, 8, 16\}$, then 8_{15} bounds a properly embedded disk $D \subset S^2 \times S^2 - B^4$ such that $\partial D = 8_{15}$ and $[D] = -2\alpha_1 + n\beta_1 \in H_2(S^2 \times S^2 - B^4, S^3; \mathbb{Z})$. Figure 6 ?????? shows that $\bar{6}_1 \overset{(2,2)}{\longrightarrow} K_0 \overset{(2,2)}{\longrightarrow} \bar{8}_{15}$, then there exists a properly embedded disk $\Delta \subset S^2 \times S^2 \# S^2 \times S^2 - B^4$ such that $\partial\Delta = 8_{15}$ and $[\Delta] = -2\alpha_2 + n\beta_2 + 2\alpha_3 + 2\beta_3 \in H_2(S^2 \times S^2 \# S^2 \times S^2, S^3; \mathbb{Z})$. The sphere

$$[S] = [D \cup \Delta] = -2\alpha_1 + 2\beta_1 + 2\alpha_2 + n\beta_2 + 2\alpha_3 + 2\beta_3 \in H_2(3S^2 \times S^2, \mathbb{Z})$$

is a characteristic class. This would contradict Theorem 2.4.

Now assume that $n = 4$, then 8_{15} bounds a properly embedded disk $D \subset S^2 \times S^2 - B^4$ such that

$$[D] = -2\alpha_1 + 4\beta_1 \in H_2(S^2 \times S^2 - B^4, S^3; \mathbb{Z}),$$

and $\partial D = 8_{15}$. Since the 4-ball genus of 8_{15} is two, then consider the orientable and compact surface $(\Sigma_2, \partial\Sigma_2) \subset (B^4, \partial B^4 \cong S^3)$ such that $\partial\Sigma_2 = \bar{8}_{15}$. Gluing Δ and Σ_2 along their boundaries yield a closed surface $\Sigma = \Delta \cup \Sigma_2 \subset S^2 \times S^2$ representing $-2\alpha + 4\beta \in H_2(S^2 \times S^2 - B^4)$, whose genus is also two (see Figure 5). This would contradict Theorem 2.5 since $G_{min}(-2\alpha + 4\beta) = (2-1)(4-1) = 3$.

Assume now that ω is odd, then let $d > 2$ denote the smallest prime divisor of ω. Theorem 2.2 yields that $| -n\omega^2 \frac{d^2-1}{2d^2} - \sigma_d(8_{15}) - (-n) | \leq n.$

$$| -n(\frac{\omega^2}{d^2})\frac{d^2-1}{2} + 4 + n | \leq n,$$

or equivalently,

$$| n[(\frac{\omega^2}{d^2})\frac{d^2-1}{2} - 1] - 4 | \leq n$$

Then the only possibilities are $\omega = 3$ and $n = 1$ or 2.

If $\omega = 3$ and $n = 1$, then using a gluing argument as above, we have a characteristic sphere in $\overline{\mathbb{C}P^2} \# S^2 \times S^2$ with homology class $3\bar{\gamma} - 2\alpha + 2\beta - 2\alpha' + 2\beta'$. This would contradict Theorem 2.4.

If $\omega = 3$ and $n = 2$, then 8_{15} bounds a properly embedded disk $D \subset S^2 \times S^2 - B^4$ such that

$$[D] = -3\alpha_1 + 3\beta_1 \in H_2(S^2 \times S^2 - B^4, S^3; \mathbb{Z}),$$

and $\partial D = 8_{15}$. This would contradict Theorem 2.2.

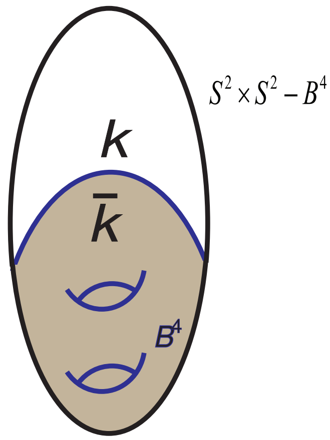

Figure 3:

$$| \xi \frac{d^2-1}{2d^2} - \sigma_d(8_{15}) - \sigma_d(S^2 \times S^2) | \leq n$$

$$| 2(-3)(3). \frac{3^2-1}{2(3)^2} - (-4) - 0 | \leq 2$$

$$| -8 + 4 | \leq 2$$

$$| -4 | \leq 2$$

$$4 \leq 2$$

Case II.$n < 0$

In this case, by Theorem 2.2, we can conclude that either $\omega = 0$ or $\omega = 1$. If $n < 0$ and $\omega = 0$, then 8_{15} bounds a properly embedded disk $D \subset S^2 \times S^2 - B^4$ such that $[\Delta] = 0 \in H_2(S^2 \times S^2 - B^4, S^3, \mathbb{Z})$ and $\partial\Delta = 8_{15}$.

Figure 6 ?????? shows that $\bar{6}_1 \xrightarrow{(2,2)} K_0 \xrightarrow{(2,2)} \bar{8}_{15}$, then there exists a properly embedded disk $D \subset S^2 \times S^2 \# S^2 \times S^2 - B^4$ such that $\partial\Delta = 8_{15}$ and $[D] = -2\alpha_1 + 2\beta_1 - 2\alpha_2 + 2\beta_2 \in H_2(S^2 \times S^2 \# S^2 \times S^2 - B^4, S^3; \mathbb{Z})$. The sphere

$$[S] = [D \cup \Delta] = -2\alpha_1 + 2\beta_1 - 2\alpha_2 + 2\beta_2 \in H_2(3S^2 \times S^2, \mathbb{Z})$$

is a characteristic class. This would contradict Theorem 2.4. Therefore, the only remaining case to preclude is $n < 0$ and $\omega = 1$. We denote $U \overset{(-n,1)}{\to} \overline{8}_{15}$. If $-n = |n| \geq 2$, then by Theorem 2.1,

$\triangle_{\bar{8}_{15}}(t) = \pm t^r$ $(mod. \quad (1-t)\sigma_n(t))$, or equivalently,

$\triangle_{\bar{8}_{15}}(t) \mp t^r$ is a multiple of $(1-t^n)$. Since $\triangle_{\bar{8}_{15}}(1) = \triangle_{\bar{8}_{15}}(t) - t^r = (1-t^n)Q(t)$ for some $r \in Z$, then n is odd. Let $\triangle_{\bar{8}_{15}}(t) - t^r = (t^n - 1)Q(t)$, where

$$Q(t) = a_q t^q + a_{q-1}t^{q-1} + ...a_1 t + a_0.$$

Therefore,

$\triangle_{\bar{8}_{15}}(t) - t^r = a_q t^{n+q} + a_{q-1}t^{n+q-1} + ... + a_1 t^{n+1} + a_0 t^n - a_q t^q - ... - a_0$. Assume that $r > 4$, then $a_q = -1$ and $a_0 = -3$. Hence, $\triangle_{\bar{8}_{15}}(t) = a_{q-1}t^{n+q-1} + ... + a_1 t^{n+1} + a_0 t^n - a_q t^q - ... - a_1 t + 3$.

Denote by $a_{q-\ell}$ the first term $a_{q-\ell} \neq 0$ and $a_{q-i} = 0$ for $1 \leq i \leq \ell$, then $\triangle_{\bar{8}_{15}}(t) = a_{q-\ell}t^{n+q-\ell} + ... + a_1 t^{n+1} - 3t^n - a_q t^q - ... - a_1 t + 3$. ($0 \leq \ell \leq q$). If $a_{q-\ell}t^{n+q-\ell} = a_{q-j}t^{q-j}$, then by induction we go to the next. Therefore, without loss of generality, we can assume that the degree of $\triangle_{\bar{8}_{15}}(t) = n + q - 1$, which implies that $n \leq 4$. Since n must be odd, then $n = -3$. The case $n = -1$ can be easily excluded by Kikuchi's theorem. To exclude $n = -3$, we consider the complex root of $t^3 - 1$, which is $t = e^{i\frac{4\pi}{3}}$. Therefore,

$\triangle_{\bar{8}_{15}}(e^{i\frac{4\pi}{3}}) = e^{i\frac{4r\pi}{3}}$. By expanding, we have:

$$3 - 8e^{i\frac{4\pi}{3}} + 11e^{i\frac{8\pi}{3}} - 8e^{i\frac{12\pi}{3}} + 3e^{i\frac{16\pi}{3}} = e^{i\frac{4r\pi}{3}}$$

This implies that,

$$3 - 8cos(\frac{4\pi}{3}) + 11cos(\frac{8\pi}{3}) - 8cos(\frac{12\pi}{3}) + 3cos(\frac{16\pi}{3}) = cos(\frac{4r\pi}{3})$$

Or equivalently,

$$cos(\frac{4r\pi}{3}) = -8,$$

a contradiction.

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