Smallest non-twisted knots

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We prove that the $(-1,2)$-cable of the right-handed trefoil knot, depicted in Figure 1 (a), is the smallest non-twisted prime satellite knot and that 7_5, depicted in Figure 1 (b), is the smallest non-twisted hyperbolic knot. We also give a classification of twisting of knots up to seven crossings.

57M25; 57Q45.

1 Introduction

Let K be a knot. Recall that the (p,q)-cable of K, denoted $K_{p,q}$, is a satellite knot with pattern the (p,q)-torus knot, $T(p,q)$. In other words, $K_{p,q}$ is a knot drawn on the boundary of a tubular neighborhood $\partial N(K)$ of K, with slope $\dfrac{p}{q}$ with respect to the standard framing of this torus. Throughout, we will assume $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$ (see Chapter $4, D$ of Rolfsen's book [32] for more details on cable knots).

Example 1.1. In this paper, $T(2,3)_{\pm 1,2}$ denotes the $(\pm 1,2)$-cable of the right-handed trefoil knot (see Figures 1 and 3), in which case $K \cong T(2,3)$ and $(p,q) = (\pm 1,2)$.

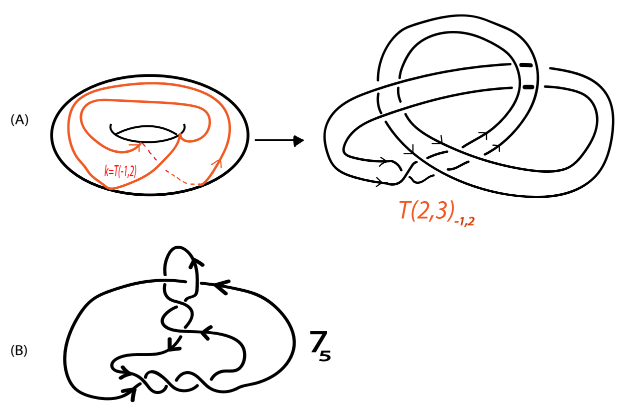


Figure 1

Let K be an unknot in the 3-sphere S^3, and D a disk in S^3 meeting K transversely more than once in the interior (see Figure 2). We assume that $\mid D \cap K \mid$ is minimal and greater than one over all isotopies of K in $S^3 - \partial D$. We call such a disk D a *twisting disk* for K. Let n be an integer and $\omega = lk(\partial D^2, K)$. Let K_n be a knot in S^3 obtained by n twisting along the disk D, in other words, $(-1/n)$-Dehn surgery along $C = \partial D$. We say that K_n is obtained from K by (n, ω)*-twisting*. Then we write $K \stackrel{(n,\omega)}{\rightarrow} K_n$. Sometimes we use the terminology that K_n is a (n, ω)*-twisted knot*.

Example 1.1. Figure 3 proves that the positive prime satellite knot $T(2, 3)_{1,2}$ is $(+1, 4)$-twisted.

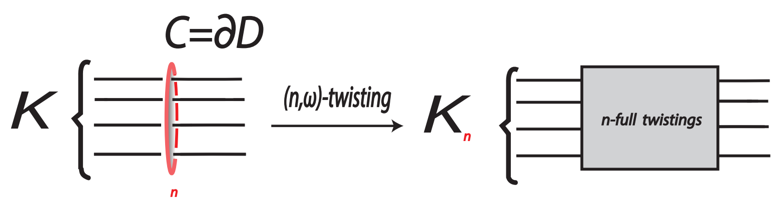


Figure 2

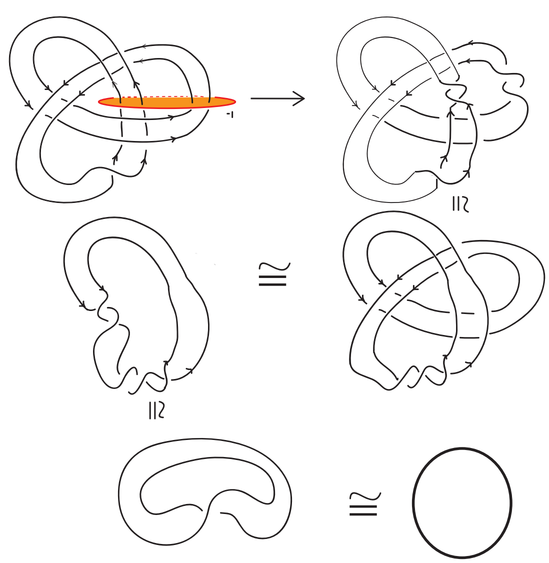


Figure 3

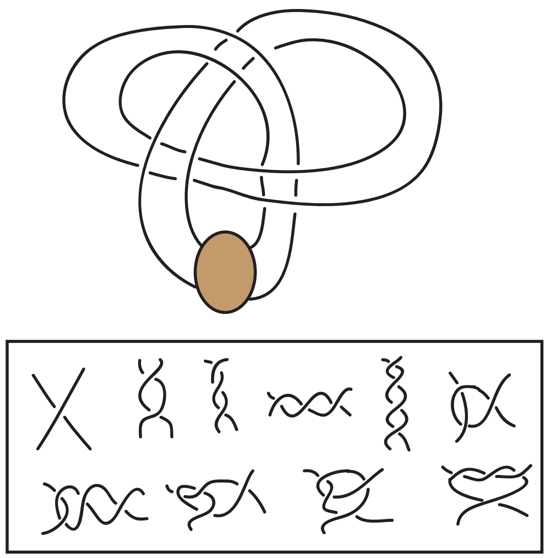


Figure 4

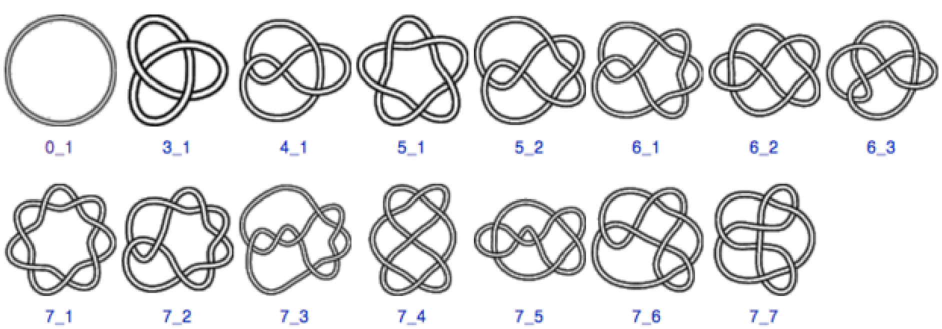


Figure 5

Active research in twisting of knots started around 1990. One pioneer was the authors' thesis advisor, Y. Mathieu [24], who investigated the general question of whether knots are twisted or not. Y. Ohyama [29] showed that any knot can be untied by (at most) two disks. K. Miyazaki and A.Yasuhara [26] were the first to give an infinite family of knots that are nontwisted, that is, they can not be untied by one single disk (for any $n \in \mathbb{Z}$). Furtheremore, they showed that the granny knot $T(2,3)\#T(2,3)$ is the smallest composite nontwisted knot.

Recall that by Thurston's uniformization theorem for Haken manifolds [36], the knot complement $E(k) = S^3 - \text{int } N(k)$ is either a toroidal, or a Seifert, or a hyperbolic 3-manifold. This is respectively equivalent to k is either a satellite, a torus, or a hyperbolic knot [36].

In his Ph.D. thesis [2], the author showed that the $(5,8)$-torus knot is the smallest non-twisted torus knot (see also [7]). This was followed by a joint work with A. Yasuhara [7], in which we gave an infinite family of nontwisted torus knots (i.e., $T(p, p+7)$ for any $p \geq 7$), using some gauge theory results.

J. Hoste and al. proved in [15] that each satellite knot with ≤ 16 crossings is obtained by substituting one of the tangles, or its reflection, into the shaded area as depicted in Figure 4. Then, it is easy to conclude from Figure 4 and Table $A1$ in [15] (see pages 43 and 44) that the smallest satellite knots must have 13 crossings, i.e., the $T(\pm 2, 3)_{\pm 1,2}$.

In this paper, we prove the following theorems:

Theorem 1.1 *$T(2,3)_{-1,2}$ is the smallest non-twisted prime satellite knot.*

Theorem 1.2 *7_5 is the smallest non-twisted hyperbolic knot.*

Theorem 1.2 gives a classification of twisting of knots up to seven crossings (see Figure 5) taken from [38]. More precisely, we have the following:

Corollary 1.1 *7_5 is the only nontwisted knot in the list of knots up to seven crossings.*

2 Preliminaries

To prove Theorem 1.1 and Theorem 1.2, we need subsections 2.1 through 2.6:

2.1 Embedding of surfaces in 4-manifolds

In what follows, let X be a smooth, closed, oriented and simply connected 4-manifold, then the second homology group $H_2(X;\mathbb{Z})$ is finitely generated (we refer to the book of Milnor and Stasheff [25]). The ordinary form $q_X, H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \longrightarrow \mathbb{Z}$ given by the intersection pairing for 2-cycles such that $q_X(\alpha,\beta) = \alpha \cdot \beta$, is a symmetric, unimodular bilinear form. The signature of this form, denoted $\sigma(X)$, is the difference of the numbers of positive and negative eigenvalues of a matrix representing q_X. Let $b_2^+(X)$ (resp. $b_2^-(X)$) be the rank of the positive (resp. negative) part of the intersection form of X. The second Betti number is $b_2(X) = b_2^+(X) + b_2^-(X)$, and the signature is $\sigma(X) = b_2^+(X) - b_2^-(X)$. From now on, a homology class in $H_2(X - B^4, \partial(X - B^4);\mathbb{Z})$ is identified with its image by the homomorphism

$$H_2(X - B^4, \partial(X - B^4);\mathbb{Z}) \cong H_2(X - B^4;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z})$$

Recall that $\mathbb{CP}^2 = (\mathbb{C}^3 - \{(0,0,0)\})/\mathbb{C}^*$ i.e. $\mathbb{CP}^2$ is the 4-manifold obtained by the free action of $\mathbb{C}^* = \mathbb{C} - \{0\}$ on $\mathbb{C}^3 - \{(0,0,0)\}$ defined by $\lambda(x,y,z) = (\lambda x, \lambda y, \lambda z)$ where $\lambda \in \mathbb{C}^*$. An element of $\mathbb{CP}^2$ is denoted by its homogeneous coordinates $[x,y,z]$, which are defined up to the multiplication by $\lambda \in \mathbb{C}^*$. The fundamental class of the submanifold $H = \{[x,y,z] \in \mathbb{CP}^2 | x = 0\} (H \cong \mathbb{CP}^1)$ generates the second homology group $H_2(\mathbb{CP}^2;\mathbb{Z})$ (see R. E. Gompf and A.I. Stipsicz [11]). Since $H \cong \mathbb{CP}^1$, then the standard generator of $H_2(\mathbb{CP}^2;\mathbb{Z})$ is denoted by $\gamma = [\mathbb{CP}^1]$. Therefore, the standard generator of $H_2(\mathbb{CP}^2 - B^4;\mathbb{Z})$ is $\mathbb{CP}^1 - B^2 \subset \mathbb{CP}^2 - B^4$ with the complex orientations.

A second homology class $\xi \in H_2(X;\mathbb{Z})$ is said to be characteristic if ξ is dual to the second Stiefel-Whitney class $w_2(X)$; or equivalently $\xi \cdot x \equiv x \cdot x \pmod{2}$ for any $x \in H_2(X;\mathbb{Z})$ (we leave the details to Milnor and Stasheff book [25]).

Example 2.1. $(a,b) \in H_2(S^2 \times S^2;\mathbb{Z})$ is characteristic iff a and b are both even.

Example 2.2. $d\gamma \in H_2(\mathbb{CP}^2;\mathbb{Z})$ $\left(\text{resp. } d\bar{\gamma} \in H_2(\overline{\mathbb{CP}^2};\mathbb{Z})\right)$ is characteristic iff d is odd.

The following theorem is originally due to O.Ya.Viro [40]. It is also obtained by letting $a = [p/2]$ in the inequality of [10, Remarks(a) on p-371] by P. Gilmer. In what follows, $\sigma_p(k)$ denotes the Tristram's p-signature of a knot k [39].

Theorem 2.1 *Let X be an oriented and compact 4-manifold with ∂X is the 3-sphere, and K a knot in ∂X. Suppose K bounds a surface of genus g in X representing an element ξ in $H_2(X;\partial X)$.*

(1) *If ξ is divisible by an odd prime p, then* $\left|\frac{p^2-1}{2p^2}\xi^2 - \sigma(X) - \sigma_p(K)\right| \leq dim\, H_2(X;\mathbb{Z}_p) + 2g.$

(2) *If ξ is divisible by 2, then* $\left|\frac{\xi^2}{2} - \sigma(X) - \sigma(K)\right| \leq dim\, H_2(X;\mathbb{Z}_2) + 2g.$

Theorem 2.2 *(Rohlin [34]) Let X be an oriented and compact 4-manifold. Suppose Σ is an embedded smooth, closed and oriented sphere in X and denote $\xi = [\Sigma] \in H_2(X, \partial X)$. Then*

(1) *If ξ is divisible by an odd prime p, then* $\left|\frac{p^2-1}{2p^2}\xi^2 - \sigma(X)\right| \leq dim\, H_2(X;\mathbb{Z}_p).$

(2) *If ξ is divisible by 2, then* $\left|\frac{\xi^2}{2} - \sigma(X)\right| \leq dim\, H_2(X;\mathbb{Z}_2).$

Theorem 2.3 *(Kikuchi [20]) Let X be a closed, oriented and simply connected 4-manifold such that (1) $H_1(X;\mathbb{Z})$ has no 2-torsion and (2) $0 \leq b_2^{\pm}(X) \leq 3$. Let ξ be a characteristic element of $H_2(X;\mathbb{Z})$. If ξ is represented by a 2-sphere, then*

$$\xi^2 = \sigma(X).$$

Theorem 2.4 *(Acosta [1]) Suppose that ξ is a characteristic homology class in an indefinite smooth oriented 4-manifold of genus g. Let $m = min(b_2^+(X), b_2^-(X))$.*

(1) *If $\xi^2 \equiv \sigma(X)$ $(mod\ 16)$, then either $\xi^2 = \sigma(X)$ or, if not,*

   (a) *If $\xi^2 = 0$ or ξ^2 and $\sigma(X)$ have the same sign, then* $\frac{\left|\xi^2 - \sigma(X)\right|}{8} \leq m + g - 1.$

   (b) *If $\sigma(X) = 0$ or ξ^2 and $\sigma(X)$ have opposite signs, then* $\frac{\left|\xi^2 - \sigma(X)\right|}{8} \leq m + g - 2.$

(2) *If $\xi^2 \equiv \sigma(X) + 8$ $(mod\ 16)$, then*

   (a) *If $\xi^2 = -8$ or ξ^2+8 and $\sigma(X)$ have the same sign, then* $\frac{\left|\xi^2 + 8 - \sigma(X)\right|}{8} \leq m + g + 1.$

   (b) *If $\sigma(X) = 0$ or ξ^2+8 and $\sigma(X)$ have opposite signs, then* $\frac{\left|\xi^2 + 8 - \sigma(X)\right|}{8} \leq m + g.$

The following theorem is the definition of Robertello's Arf invariant:

Theorem 2.5 *(Robertello [31]) Let X be an oriented and compact 4-manifold with ∂X is the 3-sphere, and K a knot in ∂X. Suppose K bounds a disk in X representing a characteristic element ξ in $H_2(X;\partial X)$, then* $\frac{\xi^2 - \sigma(X)}{8} \equiv Arf(K)\ (mod\ 8).$

2.2 The minimal genus problem in $S^2 \times S^2$ (Ruberman [33])

The genus function G is defined on $H_2(X;\mathbb{Z})$ as follows: For $\alpha \in H_2(X,\mathbb{Z})$, consider

$$G(\alpha) = min\{genus(\Sigma) | \Sigma \subset X \quad represents \quad \alpha, i.e., [\Sigma] = \alpha\}$$

where Σ ranges over closed, connected, oriented surfaces smoothly embedded in the 4-manifold X. Note that $G(-\alpha) = G(\alpha)$ and $G(\alpha) \geq 0$ for all $\alpha \in H_2(X;\mathbb{Z})$ (we leave the details to Gompf and Stipsicz [11] and Lawson [23]).

Theorem 2.6 *(Ruberman) Let $\alpha = [S^2 \times \{pt.\}]$ and $\beta = [\{pt.\} \times S^2]$ be the standard generators of $H_2(S^2 \times S^2;\mathbb{Z})$ with $\alpha \cdot \alpha = \beta \cdot \beta = 0$ and $\alpha \cdot \beta = 1$. If $ab \neq 0$ then*

$$G(a\alpha + b\beta) = (\mid a \mid -1)(\mid b \mid -1).$$

Obviously $G(a\alpha) = G(b\beta) = 0$.

Proposition 2.1 *Let g^* is the 4-ball genus of k and assume that k be a (n, ω)-twisted knot with n is even and $\omega \neq 0$, then*

$$(\mid \omega \mid -1)(\frac{\mid n\omega \mid}{2} - 1) \leq g^*,$$

Proof Assume that a knot k is (n, ω)-twisted and assume that n is even and $\omega \neq 0$. Then k bounds a disk $(D, \partial D) \subset \left(S^2 \times S^2 - B^4, \partial(S^2 \times S^2 - B^4) \cong S^3\right)$ such that:

$$\partial D = k \text{ and } [D] = -\epsilon\omega\alpha + \frac{\mid n \mid \omega}{2}\beta \in H_2(S^2 \times S^2 - B^4, S^3;\mathbb{Z}); \text{ with } \epsilon = sign(n)$$

(See Lemma 3.2 in [7], or K. Miyazaki and A. Yasuhara [26], Fig. 4 on p-146 as well as Cochran and Gompf [8], Fig. 12 on p-506).

Let $(S_{g^*}, \partial S_{g^*}) \subset (B^4, \partial B^4 \cong S^3)$ be a compact, connected and oriented surface such that $\partial S_{g^*} = \bar{k}$, where $\bar{k} = -k^*$ is the dual knot of k, that is, the inverse of the mirror image of k [19]. Gluing Δ and S_{g^*} along their boundaries k yields a smooth closed genus g^* surface $\Sigma_{g^*} = \Delta \bigcup_k S_{g^*}$ embedded in $S^2 \times S^2$. By Theorem 2.6 we have,

$$G(\pm\omega\alpha + \frac{n\omega}{2}\beta) = (\mid \omega \mid -1)(\frac{\mid n\omega \mid}{2} - 1).$$

Therefore, $(\mid \omega \mid -1)(\frac{\mid n\omega \mid}{2} - 1) \leq g^*$. □

As a corrolary of Theorem 2.6 and Proposition 2.1 we have:

Corollary 2.1 *Assume that k that is (n, ω)-twisted with $g^* = 1$, where n and ω are both even, then $\omega = 0$ or $(n, \omega) = (\pm 2, \pm 2)$.*

2.3 Congruence classes of Knots (Nakanishi and Suzuki [28])

The notion of congruence classes of knots - due to R. H. Fox [9] - is an equivalence relation generated by certain twistings. A necessary condition for congruence is given by Nakanishi and Suzuki [28] in terms of Alexander polynomials.

Definition 2.1 *[28]*

(1) *Let n, ω be non-negative integers. We say that a knot K is ω-congruent to a knot L modulo n, ω and write $K \equiv L \pmod{n, \omega}$ if there is a sequence of knots $K = K_1, K_2, ..., K_m = L$ such that for each $i \in \{1, ..., m-1\}$, K_{i+1} is obtained from K_i by some (n_i, ω)-twisting, where $n_i \equiv 0 \pmod n$ and $\omega_i \equiv 0 \pmod \omega$*

(2) *If $\omega_i = \omega$ for all $i \in \{1, ..., m-1\}$, then we say K is ω-congruent to a knot L modulo n, and we write $K \equiv^{\omega} L$ (mod. n).*

Theorem 2.7 *(Nakanishi and Suzuki [28]) If $K \equiv L \pmod{n, \omega}$ then*

(1) *$\Delta_K(t) \pm t^r \Delta_L(t)$ is a multiple of $(1-t)\sigma_n(t^{\omega})$ for some integer r, where $\sigma_n(t) = \dfrac{t^n - 1}{t - 1}$.*

(2) *If n or ω is even, then $\Delta_K(-1) \equiv \Delta_L(-1) \pmod{2n}$.*

Example 2.3. Figure 6 shows that $7_5 \equiv^2 U$ (mod. 2), where U is the unknot.

Remark 2.1 *Let $n \geq 1$ and $\omega \geq 1$. If $U \xrightarrow{(n,\omega)} K_n$ then $K_n \equiv^{\omega} U$ (mod. 2). In particular, if n or ω is even, then n divides $\dfrac{det(K) - 1}{2}$, where $det(K) = \Delta_K(-1)$.*

2.4 Tristram's signatures of satellite knots [22, 39]

Let K be a knot, M a Seifert matrix for K and ξ a complex number of modulus 1, that is, $\xi = e^{2i\pi x}$ for $0 \leq x \leq 1$. Denote by $\sigma_{\xi}(K)$ the signature of the Hermitian matrix $V(\xi) = (1-\xi)M + (1-\bar{\xi})M^t$. The signature of a knot is $\sigma(K) = \sigma_{-1}(K)$ and the Tristram p-signature ($p \geq 3$ and prime) corresponds to $x = \dfrac{p-1}{p}$ [39]; in which case $\dfrac{2\pi}{3} \leq 2\pi x \leq \pi$. The matrix $V(\xi)$ is singular if and only if ξ is a root of the Alexander polynomial $\Delta(t)$ of K. The signature of $V(z)$ for $z \in S^1$ is continuous at $z = z_0$ if $V(z_0)$ is a nonsingular matrix. Thus, if the arguments of the roots of $\Delta(t)$ do not lie in $[2\pi/3, \pi]$, then Tristram's p-signatures of K do not depend on p.

The signatures of a satellite knot are determined by those of its constituent parts. We suppose given an unknotted solid torus $V \subset S^3$ and a knot k contained with (algebraic) winding number q in the interior of V. From this "pattern" and any knot K we construct a satellite knot $K^{\star}$ by taking a faithful embedding $f : V \mapsto S^3$ with f (core of V) $= K$, and setting $K^{\star} = f(k)$.

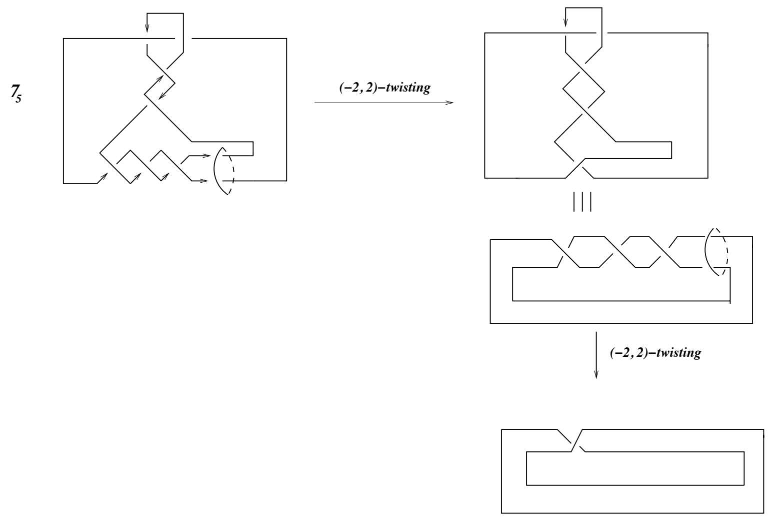


Figure 6

Theorem 2.8 *(Litherland [22]) If ξ is a root of unity,*

$$\sigma_\xi(K^\star) = \sigma_{\xi^q}(K) + \sigma_\xi(k)$$

Lemma 2.1 *The Tristram's p-signatures of $T(2,3)_{-1,2}$ and 7_5 are respectively given as follows:*

(1) $\sigma_p(T(2,3)_{-1,2}) = -2$, *and*

(2) $\sigma_p(7_5) = -4$.

Proof (1) To prove that $\sigma_d(T(2,3)_{-1,2}) = -2$, we use Theorem 2.8. Indeed, Figure 1 shows that the pattern k is the unknot, $q = +2$, $K = T(2,3)$ and $\xi = e^{2i\pi\frac{(p-1)}{p}}$. Thus, $\sigma_p(T(2,3)_{-1,2}) = \sigma_{\xi^2}(T(2,3))$. The roots of $\Delta_{T(2,3)}(t) = t^2 - t + 1$ are $t = e^{\pm i\frac{\pi}{3}}$ whose arguments do not lie in $[2\pi/3, \pi]$, then Tristram's p-signatures of $T(2,3)$ do not depend on p. Since $\sigma(T(2,3)) = -2$, therefore $\sigma_p(T(2,3)_{-1,2}) = -2$.

(2) Similarly, the roots of $\Delta_{7_5}(t) = 2 - 4t + 5t^2 - 4t^3 + 2t^4$ are

$$t \in \{0.14645 \pm 0.98922i, 0.85355 \pm 0.52101i\}.$$

The respective arguments are (see [37])

$$\theta \in \{\pm 0.54803411475747, \pm 1.4238179906773\} \cong \{\pm 0.17\pi, \pm 0.45\pi.\}$$

These arguments do not lie in $[2\pi/3, \pi]$, then Tristram's p -signatures of K do not depend on p. Since $\sigma(7_5) = -4$, therefore $\sigma_p(7_5) = -4$.

$\square$

2.5 Twisted Graph knots

As $M = S^3 - \text{int } N(K \cup C)$ is a Haken manifold, then M has a ***torus decomposition*** in the sense of W. Jaco and P.B. Shalen [16, 17], and K. Johannson [18]: There exists a finite family of decomposing essential tori $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \ldots \cup \mathcal{T}_n$ (Possibly $\mathcal{T} = \emptyset$), such that each piece of $E(k)$ is either a Seifert fibred space or a simple manifold. Each piece is either a hyperbolic space (H), or a torus space (T), or a cabling space (Ca), or a decomposing space (Co) [16, 17].

Definition 2.2 *A knot in S^3 is called a graph knot if its exterior is a graph manifold, i.e., there is a family of tori which decompose the exterior into Seifert fiber spaces.*

Note that a graph knot results from the unknot K_0 by cabling and connected sum operations,

$$K_0 \xrightarrow{cabling} K_1 \xrightarrow[connected \quad sum]{cabling} K_2 \cdots \xrightarrow[connected \quad sum]{cabling} K_m = k.$$

By the uniqueness of the solid torus decomposition, we may assume from now on that any companion of a graph knot is either a torus knot, or a composite knot or a cable knot. If every companion of a graph knot k is an exceptional torus knot, i.e. , of the form $T(p, np \pm 1)$ for $(n, p) \in \mathbb{Z} \times \mathbb{Z}$, then k is called exceptional. Otherwise, k is a non-exceptional graph knot.

Definition (Exceptional pair). Let K^0 be a trivial knot intersecting a disk D exactly once; $K \cup \partial D$ be a Hopf link in S^3. We define K^i to be an (ϵ_i, q_i)-cable of K^{i-1} for $(1 \leq i \leq m)$, i.e., K^i is an essential, simple closed curve on the boundary of a small tubular neighborhood of K^{i-1} wrapping ϵ_i (respectively q_i) times in meridional (respectively longitudinal) direction, where $\epsilon_i = \pm 1$ and $q_i \geq 2$. Then K^m is a trivial knot in S^3 and $K^m_{D,n}$ is an iterated torus knot for any integers m and n; in particular, $K^1_{D,n}$ is an $(\epsilon_1 + nq_1, q_1)$-torus knot $T(\epsilon_1 + nq_1, q_1)$ and if further $q_1 = 2$ then $K^1_{D,-\epsilon_1}$ is a trivial knot, see Fig. 1 in which $m = 1$. A pair (K, D) is called an exceptional pair of type $(\epsilon_1, q_1; \cdots; \epsilon_m, q_m)$ if the link $K \cup \partial D$ is isotopic to a link $K^m \cup \partial D$ for some integer m. In this paper we will need the following theorem:

Theorem 2.9 *(Aït Nouh-Matignon-Motegi [5]) Suppose that K is a trivial knot and D a twisting disk for K. If a knot $K_{D,n}$ is a graph knot, then $\mid n \mid= 1$ or (K, D) is an exceptional pair.*

If (K, D) is an exceptional pair, then $K_{D,n}$ is an iterated torus knot with Gromov volume $\| K_{D,n} \|= 0$ for any integer n. For the definition of Gromov volumes, see [13], [[36], Section 6], [35].

2.6 Classification of Exceptional Graph Knots:

For any twisting pair (K,D), the exterior $S^3-\text{int } N(K\bigcup\partial D)$ is irreducible and boundary-irreducible. It follows from Thurston's uniformization theorem [27, 36] and the torus theorem [16, 18] that $S^3-\text{int } N(K\bigcup\partial D)$ is Seifert fibered, toroidal or hyperbolic. We say that a twisting pair (K,D) is Seifert fibered, toroidal or hyperbolic if $S^3-\text{int } (K\bigcup\partial D)$ is Seifert fibered, toroidal or hyperbolic, respectively. Recall the following theorem that proves that the geometric types of $S^3-\text{int } (K\bigcup\partial D)$ and $S^3-\text{int } N(K_n)$ have the same geometric type for any $\mid n\mid>1$ (see Theorem 1.2 and Proposition 1.4. in [4]).

Theorem 2.10 *(Ait Nouh-Matignon-Motegi [4]) Let (K,D) be a twisting pair and let n be an integer with $\mid n\mid>1$.*

(1) *If (K,D) is a hyperbolic pair, then $K_{D,n}$ is a hyperbolic knot.*
(2) *If (K,D) is a Seifert fibered pair, then $K_{D,n}\cong T(p,np\pm1)$ for some integer $p>0$.*
(3) *If (K,D) is a toroidal pair, then $K_{D,n}$ is a satellite knot for any integer n*

Proposition 2.2 *If $K_{D,n}$ is a twisted graph knot then $n=\pm1$ or $K_{D,n}=K^1_{D,n}\cong T(q_1,nq_1+\epsilon_1)$ or $K_{D,n}=K^2_{D,n}\cong T(\pm2,2n+\epsilon_1)_{(2n-\epsilon_1,2)}$; for any $n\in\mathbb{Z}$.*

To prove Proposition 2.2, we need the following lemma:

Lemma 2.2 (a) *If $m\geq3$ then K^m is knotted.*

(b) *If K^m is unknotted then there are two cases to distinguish according to $q_1\geq3$ or $q_1=2$. Let s denote the sign of n, i.e., $s=\dfrac{\mid n\mid}{n}$.*

(1) *If $q_1\geq3$ then $m=1$ and $K^1_{D,n}\cong T(q_1,s\mid n\mid q_1+\epsilon_1)$.*
(2) *If $q_1=2$ then $m=2$ and $K^2_{D,n}\cong T(2s,2\mid n\mid+\epsilon_1)_{(2s|n|-\epsilon_1,2)}$, i.e., $K^2_{D,n}$ is the $(2s\mid n\mid-\epsilon_1,2)$-cable of the $(2s,2\mid n\mid-\epsilon_1)$-torus knot.*

Let $P_1=P,P_2,\dots,P_m$ be decomposing pieces of $E(K\bigcup\partial D)$. By Claim 5.2 in [5], each P_i has exactly two boundary components. From Claim 5.5 in [5], $P_1\bigcup_n N(c),P_2,\dots,P_m$ are decomposing pieces of $E(K_n)=(S^3-\text{int } N(K\bigcup C))\bigcup_n N(c)$. Since K_n is a graph knot, $P_2,\dots,P_m$ are Seifert fiber spaces. Since each P_i has exactly two boundary components, P_i is a cable space. The triviality of K in S^3 implies that P_i is a (ϵ_i,q_i)-cable space, where $\epsilon_i=\pm1$ and $q_i>1$. It follows that (K,D) is an exceptional pair as desired.

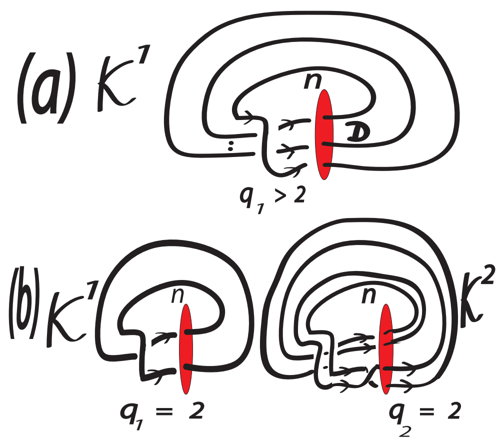


Figure 7

2.7 Twisted Positive knots

Definition 2.3 *A knot k in S^3 is called a positive knot if every crossing of k is positive.*

Example 2.4. Figure 11 shows that the knot 7_5 is a positive knot.

To prove Theorem 1.1, we need the following Proposition:

Proposition 2.3 Let k be a (n,ω)-twisted positive knot. If $n<0$ then $\omega \in \{0,\pm 1\}$. Furthermore, if $\omega = 0$ then $\dfrac{\mid \sigma(k) \mid}{2} \leq \mid n \mid$, and if $\omega = \pm 1$, then $Arf(k) = 0$.

Proof Assume that k can be obtained by (n,ω)-twisting along an unknot U. If $n<0$ then k bounds an embedded smooth disk $(D, \partial D) \subset \left(\mid n \mid \mathbb{C}P^2 - B^4, \partial\left(\mid n \mid \mathbb{C}P^2 - B^4\right) \cong S^3\right)$ such that: $[D] = \omega(\gamma_1 + ... + \gamma_{|n|}) \in H_2(\mid n \mid \mathbb{C}P^2 - B^4, S^3; \mathbb{Z})$. Here, $\mid n \mid \mathbb{C}P^2$ denotes the connected sum of $\mid n \mid$ copies of $\mathbb{C}P^2$ i.e. $\mid n \mid \mathbb{C}P^2 = \underbrace{\mathbb{C}P^2 \# \ldots \# \mathbb{C}P^2.}_{|n| \text{ times}}$

Case 1.1. If ω is even, then by Theorem 2.1

$$\left| \frac{\mid n \mid \omega^2}{2} - \mid n \mid - \sigma(k) \right| \leq \mid n \mid \Longleftrightarrow \left| \mid n \mid (\frac{\omega^2}{2} - 1) - \sigma(k) \right| \leq \mid n \mid$$

$$\Longleftrightarrow - \mid n \mid \leq \mid n \mid (\frac{\omega^2}{2} - 1) - \sigma(k) \leq \mid n \mid .$$

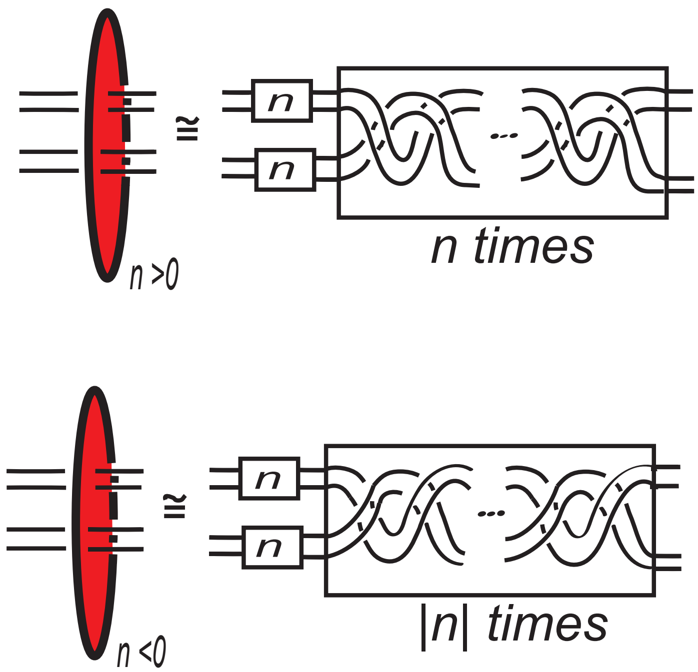
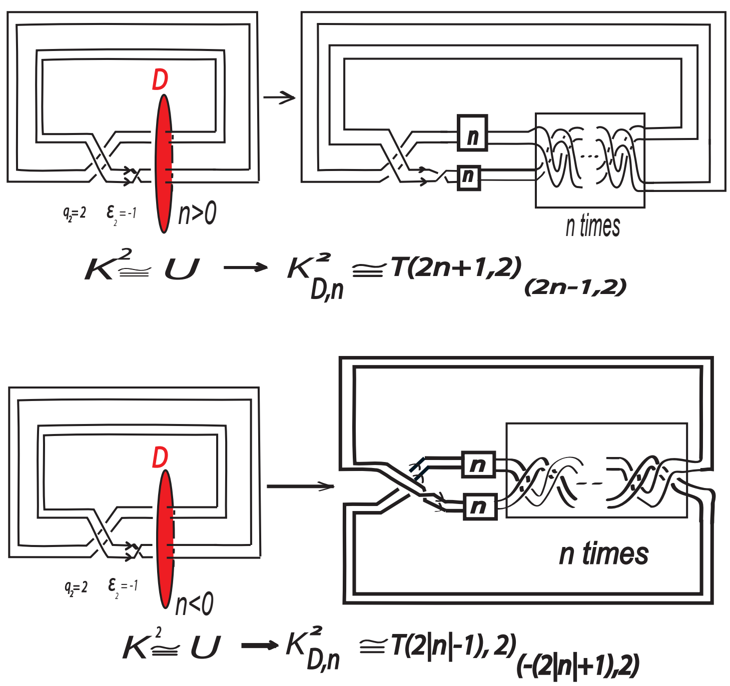


Figure 8



D
$q_2=2$ $\varepsilon_2=-1$
n>0
n
n
n times
$K^2 \cong U \longrightarrow K^2_{D,n} \cong T(2n+1,2)_{(2n-1,2)}$
D
$q_2=2$ $\varepsilon_2=-1$
n<0
n
n
n times
$K^2 \cong U \longrightarrow K^2_{D,n} \cong T(2|n|-1), 2)_{(-(2|n|+1),2)}$

Figure 9

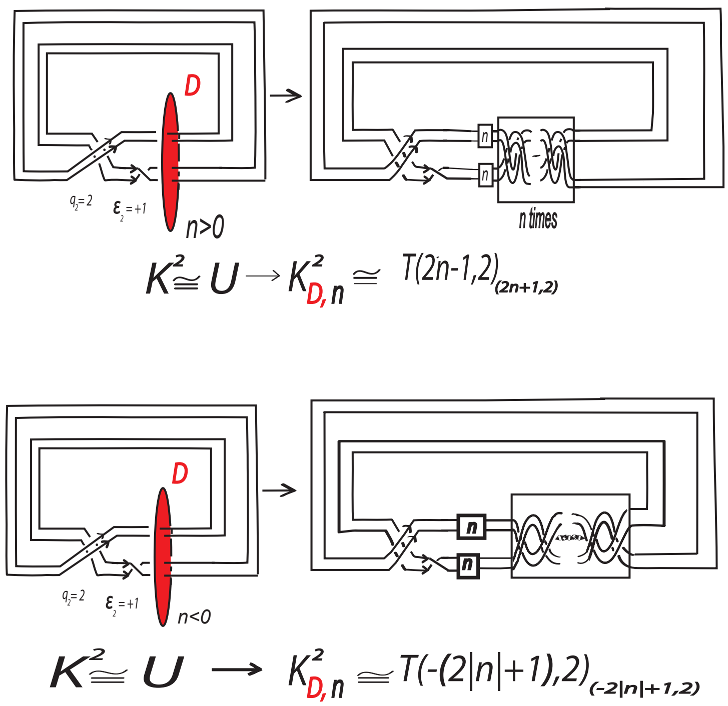


Figure 10

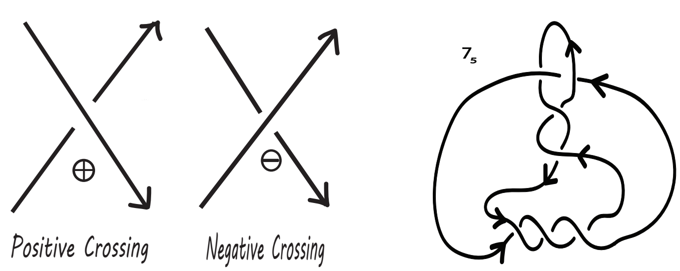


Figure 11

Since k is a positive knot, then $\sigma(k) < 0$ [30]. Therefore, the only possibility is that $\omega = 0$, which yields that $\dfrac{\mid \sigma(k) \mid}{2} \leq \mid n \mid$.

Case 1.2. If $\omega \geq 3$ is odd, then let $p > 2$ denote the smallest prime divisor of ω. By Theorem 2.1 $\left| \mid n \mid \omega^2 \dfrac{p^2-1}{2p^2} - \mid n \mid - \sigma_p(k) \right| \leq \mid n \mid \Longleftrightarrow \mid n \mid (\omega^2 \dfrac{p^2-1}{2p^2} - 2) \leq \sigma_p(k)$. Since k is a positive knot, then $\sigma_p(k) < 0$ [30]; a contradiction.

Case 1.3. If $\omega = 1$, then by Theorem 2.5, $Arf(k) = 0$.

□

3 Proof of Theorems

In what follows, we will work in the smooth category and adopt the following notations.

- U will denote the unknot in the 3-sphere.
- $\bar{k} = -k^*$ will denote the dual knot of k, that is, the inverse of the mirror image of k [19]. In particular, if $k = T(2,3)_{-1,2}$ then $\bar{k} = T(-2,3)_{+1,2}$.
- γ or γ_i for $i \in \{1, 2, \cdots, n\}$ denotes interchangeably the standard generator of $H_2(\mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$ with $\gamma \cdot \gamma = +1$ and $\gamma_i \cdot \gamma_i = +1$.
- $\bar{\gamma}$ or $\bar{\gamma}_i$ for $i \in \{1, 2, \cdot, n\}$ denotes interchangeably the standard generator of $H_2(\overline{\mathbb{CP}^2} - B^4, S^3; \mathbb{Z})$ with $\bar{\gamma} \cdot \bar{\gamma} = -1$ and $\bar{\gamma}_i \cdot \bar{\gamma}_i = -1$.
- α_i and β_i $(i = 1, 2)$ denote the standard generators of $H_2(S^2 \times S^2 - B^4, S^3; \mathbb{Z})$ with $\alpha_i = [S^2 \times \{pt_i\}]$, $\beta_i = [\{pt_i\} \times S^2]$, $\alpha_i^2 = \beta_i^2 = 0$ and $\alpha_i \cdot \beta_i = +1$.

Lemma 3.1 *$T(2,3)_{\pm 1,2}$ are the smallest satellite knots.*

Proof Hoste et al. showed in their seminal paper [15], that each satellite knot with ≤ 16 crossings is obtained by substituting one of the tangles, or its reflection, into the shaded disk as shown in Figure ??. In the other hand, they showed that the smallest satellite knot has 13 crossings and there are only two of them (see [15], Appendix I: Summary Data page 44). These two facts proves Lemma 3.1 □

3.1 Proof of Theorem 1.1.

Assume for a contradiction that $U \xrightarrow{(n,\omega)} T(2,3)_{-1,2}$. Since $T(2,3)_{-1,2}$ is a non-exceptional graph knot, by Theorem 2.9, $n = \pm 1$ [14].

Case 1 Assume for a contradiction that $n = -1$, then $T(2,3)_{-1,2}$ bounds a disk $(D, \partial D) \subset \left(\mathbb{CP}^2 - B^4, \partial(\mathbb{CP}^2 - B^4) \cong S^3\right)$ such that $[D] = \omega\gamma \in H_2(\mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$.

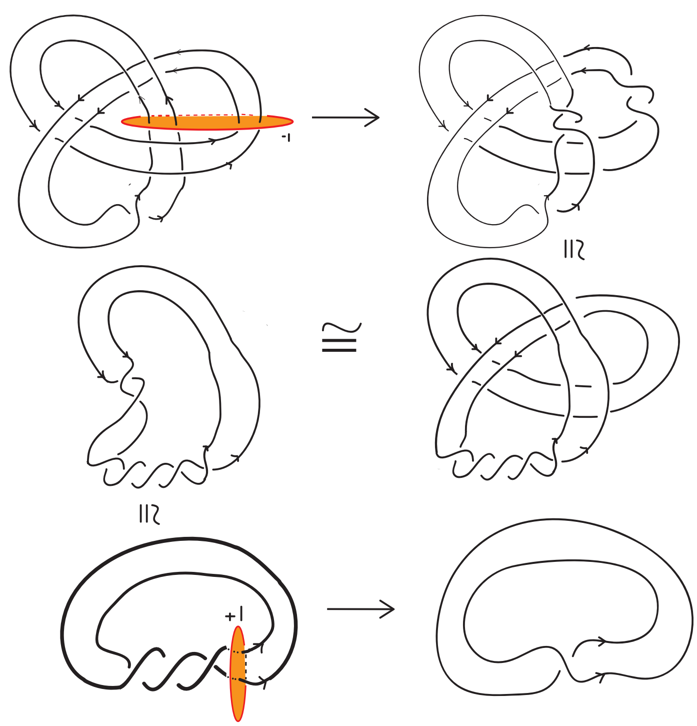


Figure 12

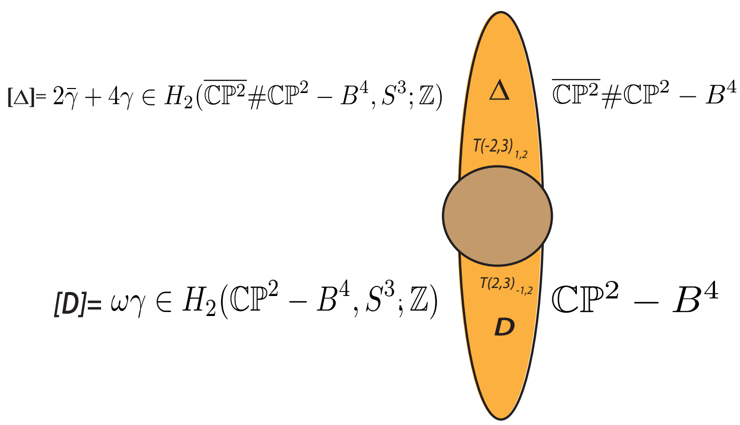

[Δ]= $2\bar{\gamma} + 4\gamma \in H_2(\overline{\mathbb{CP}^2}\#\mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$
Δ
$\overline{\mathbb{CP}^2}\#\mathbb{CP}^2 - B^4$
$T(-2,3)_{1,2}$
$T(2,3)_{-1,2}$
[D]= $\omega\gamma \in H_2(\mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$
D
$\mathbb{CP}^2 - B^4$

Figure 13

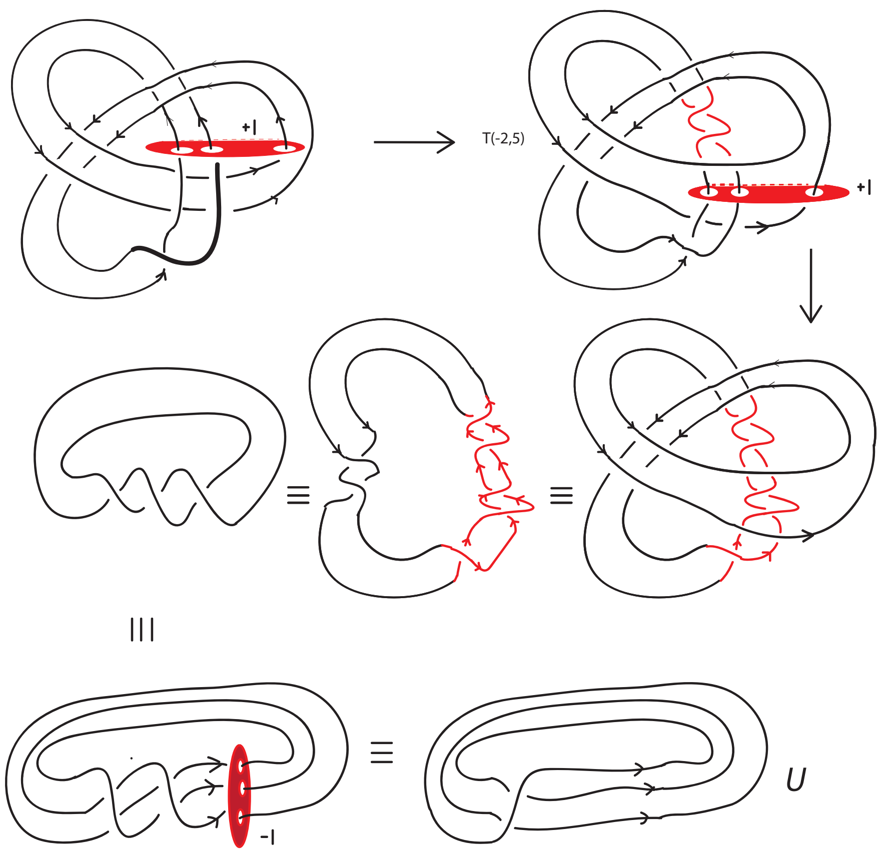


Figure 14

Case 1.1. If ω is even, then Figure 12 shows that $U \overset{(-1,2)}{\longrightarrow} T(-2,3) \overset{(+1,4)}{\longrightarrow} T(2,3)_{-1,2}$ and then, $\overline{U} \overset{(+1,2)}{\longrightarrow} T(2,3) \overset{(-1,4)}{\longrightarrow} T(-2,3)_{1,2}$. Thus, there exist a properly embedded disk $(\Delta, \partial\Delta) \subset \left(\overline{\mathbb{CP}^2}\#\mathbb{CP}^2 - B^4, \partial(\overline{\mathbb{CP}^2}\#\mathbb{CP}^2 - B^4) \cong S^3\right)$ such that $\partial\Delta = T(-2,3)_{1,2}$ and $[\Delta] = 2\bar{\gamma} + 4\gamma \in H_2(\overline{\mathbb{C}P^2}\#\mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$.

Gluing D and Δ along their boundaries, as depicted in Figure 13, would yield an embedded sphere $S \subset \mathbb{CP}^2\#\overline{\mathbb{CP}^2}\#\mathbb{CP}^2$ that represents the second homology class:

$$[S] = \omega\gamma + 2\bar{\gamma} + 4\gamma \in H_2(\mathbb{CP}^2\#\overline{\mathbb{CP}^2}\#\mathbb{CP}^2; \mathbb{Z}).$$

Since ω is even, then by Theorem 2.2, $\left|\dfrac{\omega^2+12}{2} - 1\right| \leq 3$; a contradiction.

Case 1.2. If ω is odd, then Figure 14 shows that $U \overset{(-1,3)}{\longrightarrow} T(-2,5) \overset{(-1,3)}{\longrightarrow} T(2,3)_{-1,2}$ and then, $\overline{U} \overset{(+1,3)}{\longrightarrow} T(2,5) \overset{(+1,3)}{\longrightarrow} T(-2,3)_{1,2}$. Thus, there exist a smooth disk $(\Delta, \partial\Delta) \subset \left(\overline{\mathbb{CP}^2}\#\overline{\mathbb{CP}^2} - B^4, \partial(\overline{\mathbb{CP}^2}\#\overline{\mathbb{CP}^2} - B^4) \cong S^3\right)$ such that $\partial\Delta = T(-2,3)_{-1,2}$ and $[\Delta] = 3\bar{\gamma}_2 + 3\bar{\gamma}_3 \in H_2\left(\overline{\mathbb{CP}^2}\#\overline{\mathbb{CP}^2} - B^4, \partial(\overline{\mathbb{CP}^2}\#\overline{\mathbb{CP}^2} - B^4) \cong S^3; \mathbb{Z}\right)$. Similarly, gluing D and Δ along their boundaries would yield an embedded sphere $S \subset \mathbb{CP}^2\#\overline{\mathbb{CP}^2}\#\overline{\mathbb{CP}^2}$ that represents the second homology class $[S] = \omega\gamma + 3\bar{\gamma}_1 + 3\bar{\gamma}_2 \in H_2(\mathbb{CP}^2\#\overline{\mathbb{CP}^2}\#\overline{\mathbb{CP}^2}; \mathbb{Z})$. This would contradict Theorem 2.3.

Case 2. Assume for a contradiction that $n = +1$, then $T(2,3)_{-1,2}$ bounds a smooth disk $(D, \partial D) \subset \left(\overline{\mathbb{CP}^2} - B^4, \partial(\overline{\mathbb{CP}^2} - B^4) \cong S^3\right)$ such that: $[D] = \omega\bar{\gamma} \in H_2(\overline{\mathbb{C}P^2} - B^4, S^3; \mathbb{Z})$.

Case 2.1. If ω is even, then by the same argument as in **Case 1.1**, this yields the existence of an embedded sphere $S \subset \overline{\mathbb{CP}^2}\#\overline{\mathbb{CP}^2}\#\mathbb{CP}^2$ that represents the second homology class $[S] = [D \cup \Delta] = \omega\bar{\gamma} + 2\bar{\gamma}_1 + 4\gamma \in H_2(\overline{\mathbb{CP}^2}\#\overline{\mathbb{CP}^2}\#\mathbb{CP}^2; \mathbb{Z})$. In virtue of Theorem 2.2: $\left|\dfrac{-\omega^2+12}{2} + 1\right| \leq 3$. Thus $\omega = \pm 4$. This would contradict Theorem 2.1, since by Lemma 2.1, $\sigma(T(2,3)_{-1,2}) = -2$.

Case 2.2. If ω is odd, then by the same argument as in **Case 1.2**, this would yield the existence of an embedded characteristic sphere $S \subset 3\overline{\mathbb{CP}^2}$ that represents the second homology class $[S] = [D \cup \Delta] = \omega\bar{\gamma} + 3\bar{\gamma}_2 + 3\bar{\gamma}_3 \in H_2(3\overline{\mathbb{CP}^2}; \mathbb{Z})$. This would contradict Theorem 2.3.

3.2 Proof of Theorem 1.2.

Figure 15 shows that any unknotting number one knot is (-1)-twisted. Note that all knots with less or equal than seven crossings, except $5_1, 7_1, 7_3, 7_4$ and 7_5, are unknotting number one knots [19]; and therefore they are (-1)-twisted. In the other hand, it is easy to see that the $(2,5)$-torus knot 5_1 is $(+2,2)$-twisted and that the $(2,7)$-torus knot 7_1 is $(+3,2)$-twisted as well as Figure 16 shows that 7_3 is $(+2,2)$-twisted and 7_4 is $(-2,0)$-twisted. Therefore, it remains to prove that 7_5 is a nontwisted knot.

Assume for a contradiction that $U \overset{(n,\omega)}{\longrightarrow} 7_5$. Since 7_5 is a positive knot, then Proposition 2.3 implies that either (i) $n>0$, or (ii) $n<0$ and then $\omega \in \{0,\pm 1\}$. Therefore, there are two cases to distinguish according to the sign of n and the parity of ω.

Case 1. Assume for a contradiction that $n>0$. We can assume, without loss of generality, that $\omega > 0$. Therefore, by Remark 2.1, 7_5 is ω-congruent to U.

Case 1.1. If ω is even, then by Remark 2.1, n divides $\dfrac{det(7_5)-1}{2}(=8)$.

Case 1.1.1. If $n \in \{+2,+4,+8\}$ then by Corollary 2.1, either $\omega = 0$ or $(n,\omega) = (+2,+2)$. In these cases, 7_5 bounds a disk $(D,\partial D) \subset (S^2\times S^2 - B^4, S^3)$ such that $\partial D = 7_5$ and $[D] = -\omega\alpha_1 + \dfrac{n\omega}{2}\beta_1 \in H_2(S^2\times S^2 - B^4, S^3;\mathbb{Z})$. Figure 6 shows that $U \overset{(-2,2)}{\longrightarrow} T(-2,3) \overset{(-2,2)}{\longrightarrow} \bar{7}_5$. Therefore, there exist a properly embedded disk $\Delta \subset S^2\times S^2\# S^2\times S^2 - B^4$ such that $\partial\Delta = \bar{7}_5$; and

$$[\Delta] = 2\alpha_2 + 2\beta_2 + 2\alpha_3 + 2\beta_3 \in H_2(S^2\times S^2\# S^2\times S^2 - B^4, S^3;\mathbb{Z}).$$

Gluing D and Δ along their boundaries yields the existence of a characteristic sphere $S \subset 3\,S^2\times S^2$ such that:

$$[S] = [D\cup\Delta] = -\omega\alpha_1 + \frac{n\omega}{2}\beta_1 + 2\alpha_2 + 2\beta_2 + 2\alpha_3 + 2\beta_3 \in H_2(3\,S^2\times S^2;\mathbb{Z}).$$

Theorem 2.3 yields that $n\omega^2 = 16$; a contradiction.

Case 1.1.2. If $n=+1$ then 7_5 bounds a disk $(D,\partial D) \subset (\overline{\mathbb{C}P^2} - B^4, S^3)$ such that:

$$[D] = \omega\bar{\gamma} \in H_2(\overline{\mathbb{C}P^2} - B^4, S^3;\mathbb{Z}).$$

Similarly, gluing D and Δ along their boundaries yields the existence of a smooth sphere $S \subset \overline{\mathbb{CP}^2}\# S^2\times S^2\# S^2\times S^2$ such that:

$$[S] = [D\cup\Delta] = \omega\bar{\gamma} + 2\alpha_1 + 2\beta_1 + 2\alpha_2 + 2\beta_2 \in H_2(\overline{\mathbb{CP}^2}\# S^2\times S^2\# S^2\times S^2;\mathbb{Z}).$$

Theorem 2.2 implies that $\omega = \pm 4$, which in turn contradicts Theorem 2.1.

Case 1.2. If ω is odd, then 7_5 bounds a smooth embedded disk $(D,\partial D) \subset (n\overline{\mathbb{CP}^2} - B^4, S^3)$ such that: $[D] = \omega(\bar{\gamma}_1 + ... + \bar{\gamma}_n) \in H_2(n\overline{\mathbb{C}P^2} - B^4, S^3;\mathbb{Z})$. Let p be the smallest prime divisor of ω. By Lemma 2.1, $\sigma_p(7_5) = -4$. Therefore, Theorem 2.1 implies that

$$\left|\frac{\xi^2(p^2-1)}{2p^2}+4+n\right|\leq n \quad \Longleftrightarrow \quad \left|-n(\frac{\omega}{p})^2.\frac{p^2-1}{2}+4+n\right|\leq n.$$
$$\Longleftrightarrow \quad 0\leq n\left[\left(\frac{\omega}{d}\right)^2.\frac{d^2-1}{2}\right]-4\leq 2n.$$

This implies that the only remaining cases to preclude are $n\in\{+1,+2\}$ and $\omega=3$.

(1) If $n=+1$ and $\omega=3$ then this yields the existence of a characteristic sphere $S\subset\overline{\mathbb{CP}^2}\#S^2\times S^2\#S^2\times S^2$ such that:

$$[S]=[D\cup\Delta]=3\bar{\gamma}+2\alpha_1+2\beta_1+2\alpha_2+2\beta_2\in H_2(\overline{\mathbb{CP}^2}\#S^2\times S^2\#S^2\times S^2;\mathbb{Z}).$$

This would contradict Theorem 2.3.

(2) If $n=+2$ and $\omega=3$ then this yields the existence of a properly embedded disk $(D,\partial D)\subset(S^2\times S^2-B^4,S^3)$ such that:

$$\xi=[D]=-3\alpha+3\beta\in H_2(S^2\times S^2-B^4,S^3;\mathbb{Z}).$$

This would contradict Corollary 2.1.

Case 2. If $n<0$, then Proposition 2.3 yields that $\omega\in\{0,\pm1\}$ as 7_5 is a positive knot.

Case 2.1. If $\omega=0$ then by Remark 2.1.(2), n divides $\dfrac{det(7_5)-1}{2}(=8)$.

(1) If $n=-1$, then by Proposition 2.3, $\dfrac{|\sigma(7_5)|}{2}\leq|n|$; a contradiction.

(2) If $n\in\{-2,-4,-8\}$, then this yields the existence of a characteristic sphere S such that:

$$[S]=[D\cup\Delta]=2\alpha_1+2\beta_1+2\alpha_2+2\beta_2\in H_2(3S^2\times S^2,\mathbb{Z});$$

which would contradict Theorem 2.3.

Case 2.2. If $\omega=1$, then this yields the existence of a characteristic sphere,

$$[S]=[D\cup\Delta]=\gamma_1+....+\gamma_{|n|}+2\alpha_1+2\beta_1+2\alpha_2+2\beta_2\in H_2(|n|\,\mathbb{C}P^2\#2S^2\times S^2;\mathbb{Z}).$$

Note that $\xi^2=|n|+16$ and $\sigma(X)=|n|$. By Theorem 2.4

$$\left|\frac{\xi^2-\sigma(X)}{8}\right|\leq m-1;$$

which would contradict that $m=2$.

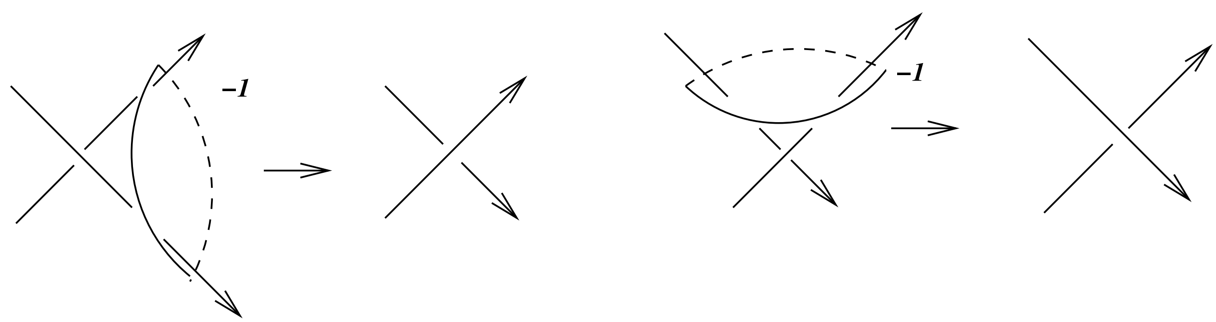


Figure 15

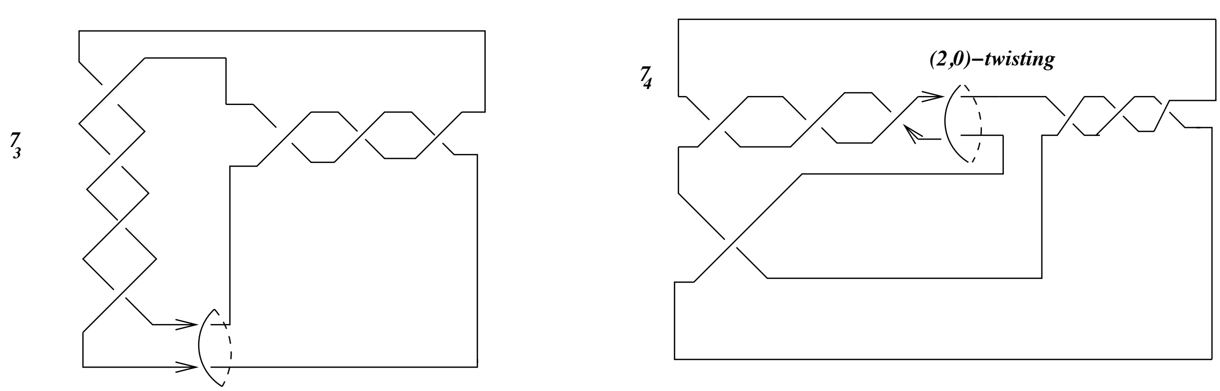


Figure 16

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