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GENERA AND DEGREES OF TORUS KNOTS IN $\mathbb{C}P^2$

MOHAMED AIT NOUH

Department of Mathematics,
University of California at Riverside,
900 University Avenue,
Riverside, CA 92521
maitnouh@math.ucr.edu

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ABSTRACT

The $\mathbb{C}P^2$-genus of a knot K is the minimal genus over all isotopy classes of smooth, compact, connected and oriented surfaces properly embedded in $\mathbb{C}P^2 - B^4$ with boundary K. We compute the $\mathbb{C}P^2$-genus and realizable degrees of $(-2, q)$-torus knots for $3 \leq q \leq 11$ and $(2, q)$-torus knots for $3 \leq q \leq 17$. The proofs use gauge theory and twisting operations on knots.

Keywords: Smooth genus; $\mathbb{C}P^2$-genus; twisting; blow-up.

Mathematics Subject Classification 2000: 57M25, 57M27

1. Introduction

Throughout this paper, we work in the smooth category. All orientable manifolds will be assumed to be oriented unless otherwise stated. A knot is a smooth embedding of S^1 into the 3-sphere $S^3 \cong \mathbb{R}^3 \cup \{\pm\infty\}$. All knots are oriented. Let K be a knot in $\partial(\mathbb{C}P^2 - B^4) \cong S^3$, where B^4 is an embedded open 4-ball in $\mathbb{C}P^2$. The $\mathbb{C}P^2$-genus of a knot K, denoted by $g_{\mathbb{C}P^2}(K)$, is the minimal genus over all isotopy classes of smooth, compact, connected and oriented surfaces properly embedded in $\mathbb{C}P^2 - B^4$ with boundary K. If K bounds a properly embedded 2-disk in $\mathbb{C}P^2 - B^4$, then K is called a slice knot in $\mathbb{C}P^2$. A similar definition could be made for any 4-manifold and that this is a generalization of the 4-ball genus.

Recall that $\mathbb{C}P^2$ is the closed 4-manifold obtained by the free action of $\mathbb{C}^* = \mathbb{C} - \{0\}$ on $\mathbb{C}^3 - \{(0,0,0)\}$ defined by $\lambda(x, y, z) = (\lambda x, \lambda y, \lambda z)$ where $\lambda \in \mathbb{C}^*$, i.e. $\mathbb{C}P^2 = (\mathbb{C}^3 - \{(0,0,0)\}/\mathbb{C}^*$. An element of $\mathbb{C}P^2$ is denoted by its homogeneous coordinates $[x : y : z]$, which are defined up to the multiplication by $\lambda \in \mathbb{C}^*$. The fundamental class of the submanifold $H = \{[x : y : z] \in \mathbb{C}P^2 | x = 0\}(H \cong \mathbb{C}P^1)$ generates the second homology group $H_2(\mathbb{C}P^2; \mathbb{Z})$ (see Gompf and Stipsicz [12]). Since $H \cong \mathbb{C}P^1$, then the standard generator of $H_2(\mathbb{C}P^2; \mathbb{Z})$ is denoted, from now on, by $\gamma = [\mathbb{C}P^1]$. Therefore, the standard generator of $H_2(\mathbb{C}P^2 - B^4; \mathbb{Z})$ is $\mathbb{C}P^1 - B^2 \subset \mathbb{C}P^2 - B^4$ with the complex orientations.

A class $\xi \in H_2(\mathbb{C}P^2 - B^4, \partial(\mathbb{C}P^2 - B^4); \mathbb{Z})$ is identified with its image by the homomorphism

$$H_2(\mathbb{C}P^2 - B^4, \partial(\mathbb{C}P^2 - B^4); \mathbb{Z}) \cong H_2(\mathbb{C}P^2 - B^4; \mathbb{Z}) \longrightarrow H_2(\mathbb{C}P^2; \mathbb{Z}).$$

Let d be an integer, then the degree-d smooth slice genus of a knot K in $\mathbb{C}P^2$ is the least integer g such that K is the boundary of a smooth, compact, connected and orientable genus g surface Σ_g properly embedded in $\mathbb{C}P^2 - B^4$ with boundary K in $\partial(\mathbb{C}P^2 - B^4)$ and degree d, i.e.

$$[\Sigma_g, \partial\Sigma_g] = d\gamma \in H_2(\mathbb{C}P^2 - B^4, \partial(\mathbb{C}P^2 - B^4); \mathbb{Z}).$$

By the above identification, we also have: $[\Sigma_g] = d\gamma \in H_2(\mathbb{C}P^2 - B^4; \mathbb{Z})$. If such a surface can be given explicitely, then we say that the degree d is *realizable*. The $\mathbb{C}P^2$-genus of a knot K, denoted by $g_{\mathbb{C}P^2}(K)$, is the minimum over these over all d.

Question 1.1. Given a realizable degree, is the underlying surface Σ_g unique, up to isotopy?

An interesting question is to find the $\mathbb{C}P^2$-genus and the realizable degree(s) of knots in $\mathbb{C}P^2$. In this paper, we compute the $\mathbb{C}P^2$-genus and realizable degrees of a finite collection of torus knots.

Theorem 1.1.

(1) $g_{\mathbb{C}P^2}(T(-2,3)) = 0$ *with realizable degree* $d \in \{\pm 2, \pm 3\}$.
(2) $g_{\mathbb{C}P^2}(T(-2,q)) = 0$ *for* $q = 5, 7$ *and* 9 *with respective realizable degrees* $\pm 3, \pm 4$ *and* ± 4.
(3) $g_{\mathbb{C}P^2}(T(-2,11)) = 1$ *with possible degree(s)* $d \in \{\pm 4, \pm 5\}$.

Note that for any $0 < p < q$, $T(p,q)$ is obtained from $T(2,3)$ by adding $(p-1)(q-1) - 2$ half-twisted bands. Then, there is a genus $\frac{(p-1)(q-1)-2}{2}$ cobordism between $T(2,3)$ and $T(p,q)$. We conjecture that the $\mathbb{C}P^2$-genus of a (p,q)-torus knot is equal to the genus of the cobordism between $T(2,3)$ and $T(p,q)$.

Conjecture 1.1. $g_{\mathbb{C}P^2}(T(p,q)) = \frac{(p-1)(q-1)}{2} - 1$.

We answer this conjecture by the positive for all $(2,q)$-torus knots with $3 \leq q \leq 17$.

Theorem 1.2.

(1) $g_{\mathbb{C}P^2}(T(2,3)) = 0$ *with realizable degree* $d = 0$.
(2) $g_{\mathbb{C}P^2}(T(2,q)) = \frac{q-3}{2}$ *for* $5 \leq q \leq 17$ *with respective possible degree*(*s*)
   - $d \in \{0, \pm 1\}$ *if* $q \in \{5, 7, 9, 11\}$, *and*
   - $d \in \{0, \pm 1, \pm 3\}$ *if* $q \in \{13, 15, 17\}$.

2. Twisting Operations and Sliceness in 4-Manifolds

Let K be a knot in the 3-sphere S^3, and D^2 a disk intersecting K in its interior. Let n be an integer. A $-\frac{1}{n}$-Dehn surgery along $C = \partial D^2$ changes K into a new knot K_n in S^3. Let $\omega = \mathrm{lk}(\partial D^2, L)$. We say that K_n is obtained from K by (n, ω)-*twisting* (or simply *twisting*). Then, we write $K \overset{(n,\omega)}{\rightarrow} K_n$, or $K \overset{(n,\omega)}{\rightarrow} K(n, \omega)$. We say that K_n is n-twisted provided that K is the unknot (see Fig. 1).

An easy example is depicted in Fig. 2, where we show that the right-handed trefoil $T(2, 3)$ is obtained from the unknot $T(2, 1)$ by a $(+1, 2)$-twisting. (In this case $n = +1$ and $\omega = +2$.)

There is a connection between twisting of knots in S^3 and dimension four: Any knot K_{-1} obtained from the unknot K (or more generally, a smooth slice knot in the 4-ball) by a $(-1, \omega)$-twisting is smoothly slice in $\mathbb{C}P^2$ with degree ω realizable by the twisting disk Δ, i.e. there exists a properly embedded smooth disk $\Delta \subset \mathbb{C}P^2 - B^4$ such that $\partial\Delta = K_{-1}$ and $[\Delta] = \omega\gamma \in H_2(\mathbb{C}P^2 - B^4, S^3, \mathbb{Z})$. For convenience of the reader, we give a sketch of a proof due to Miyazaki and Yasuhara [21]: We assume $K \cup C \subset \partial h^0 \cong S^3$, where h^0 denotes the 4-dimensional 0-handle ($h^0 \cong B^4$). The unknot K bounds a properly embedded smooth disk Δ in h^0. Then, performing a (-1)-twisting is equivalent to adding a 2-handle h^2, to h^0 along C with framing $+1$. It is known that the resulting 4-manifold $h^0 \cup h^2$ is $\mathbb{C}P^2 - B^4$ (see Kirby [18] for example). In addition, it is easy to verify that $[\Delta] = \omega\gamma \in H_2(\mathbb{C}P^2 - B^4, S^3, \mathbb{Z})$. More generally, we can prove, using Kirby calculus [18] and some twisting manipulations, that an (n, ω)-*twisted knot* in S^3 bounds a properly embedded smooth disk Δ in a punctured standard four manifold of the form $n\overline{\mathbb{C}P^2} - B^4$ if $n > 0$ (see Fig. 3), or $|\, n \,|\, \mathbb{C}P^2 - B^4$ if $n < 0$. The second homology of $[\Delta]$ can be computed from n and ω.

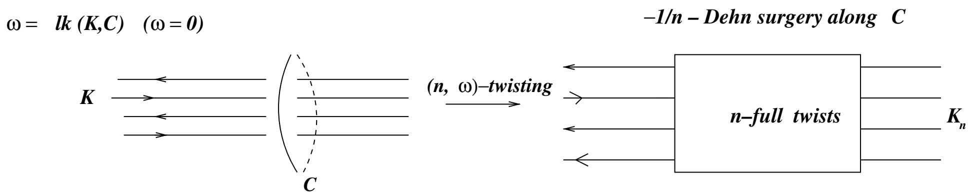

Fig. 1.

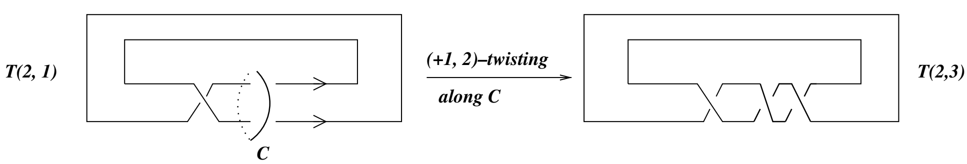

Fig. 2.

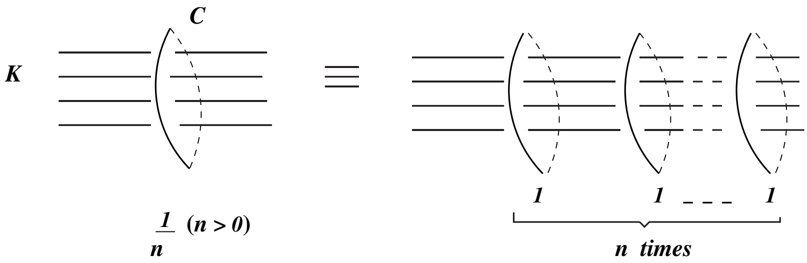

Fig. 3.

Examples.

(1) Song and Goda and Hayashi proved in [11] that $T(p, p+2)$ (for any $p \geq 5$) is obtained by a single $(+1)$-twisting along an unknot. This implies that their corresponding left-handed torus knots are smoothly slice in $\mathbb{C}P^2$ (see [2]). In [5], we proved that the realizable degree of $T(-p, p+2)$ in $\mathbb{C}P^2$ is $p+1$ (for any $p \geq 5$).

(2) Any unknotting number one knot is (-1)-twisted (see Fig. 4), and then it is smoothly slice in $\mathbb{C}P^2$. In particular, the double of any knot is smoothly slice in $\mathbb{C}P^2$.

Question 2.1. Is there a knot which is topologically but not smoothly slice in $\mathbb{C}P^2$?

The proof of Theorem 2.1 can be found in [4]:

Theorem 2.1. *If a knot K is obtained by a single (n, ω)-twisting from an unknot K_0 along C, then its inverse $-K$ is obtained by a single $(n, -\omega)$-twisting from the unknot $-K_0$ along C.*

Note that $T(-p, 4p \pm 1)$ $(p \geq 2)$ is obtained from the unknot $T(-1, 4p \pm 1)$ by a $(-1, 2p)$-twisting (see Fig. 5). Therefore, Theorem 2.2 is deduced from Kirby calculus.

Theorem 2.2. *$T(-p, 4p \pm 1)$ $(p \geq 2)$ is smoothly slice in $\mathbb{C}P^2$ with realizable degree $d = 2p$.*

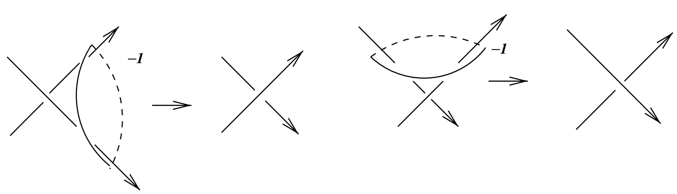

Fig. 4.

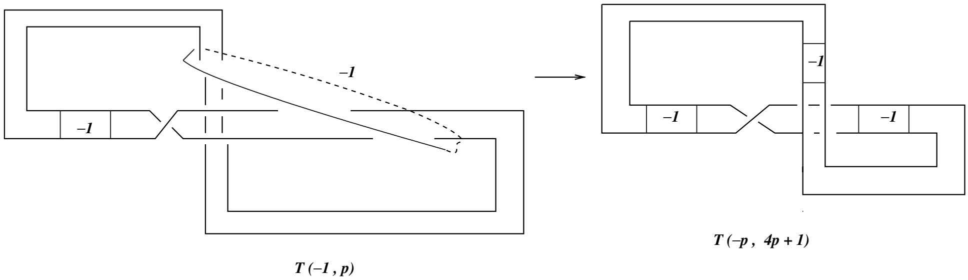

Fig. 5.

We refer the reader to my Ph.D thesis [2] for more details on twisting operations on knots in S^3.

3. Preliminaries

Litherland gave an algorithm to compute the x-signatures of torus knots.

Theorem 3.1 (Litherland [20]). *Let* $\xi = e^{2i\pi x}$, $x \in \mathbb{Q}$ *(with* $0 < x < 1$*), then*

$$\sigma_\xi(T(p,q)) = \sigma_{\xi^+} - \sigma_{\xi^-}$$

$$\sigma_{\xi^+} = \#\Big\{(i,j)|1 \le i \le p-1 \quad \textit{and} \quad 1 \le j \le q-1 \quad \textit{such that } x-1 < \frac{i}{p} + \frac{j}{q} < x \pmod 2\Big\}$$

$$\sigma_{\xi^-} = \#\Big\{(i,j)|1 \le i \le p-1 \quad \textit{and} \quad 1 \le j \le q-1 \quad \textit{such that } x < \frac{i}{p} + \frac{j}{q} < x+1 \pmod 2\Big\}$$

(i and j are integers)

If $y_{i,j} = \frac{i}{p} + \frac{j}{q}$, then $x - 1 < y_{i,j} < x \pmod 2$ is equivalent to

$$0 < y_{i,j} < x \quad \text{or} \quad x + 1 < y_{i,j} < 2.$$

The signature of a knot is $\sigma(k) = \sigma_{-1}(k)$ obtained by assigning $x = \frac{1}{2}$ and the Tristram d-signature ($d \ge 3$ and prime) corresponds to $x = \frac{d-1}{2d}$ which we denote by $\sigma_d(k) = \sigma_{e^{i\pi\frac{d-1}{d}}}$ (Tristram [24]).

In the following, $b_2^+(X)$ (respectively, $b_2^-(X)$) is the rank of the positive (respectively, negative) part of the intersection form of the oriented, smooth and compact 4-manifold X. Let $\sigma(X)$ denote the signature of M^4. Then a class $\xi \in H_2(X, \mathbb{Z})$ is said to be characteristic provided that $\xi.x \equiv x.x$ for any $x \in H_2(X, \mathbb{Z})$ where $\xi.x$ stands for the pairing of ξ and x, i.e. their Kronecker index and ξ^2 for the self-intersection of ξ in $H_2(M^4, \mathbb{Z})$.

Theorem 3.2 (Gilmer and Viro [10, 25]). *Let X be an oriented, compact 4-manifold with $\partial X = S^3$, and K a knot in ∂X. Suppose K bounds a surface of genus g in X representing an element ξ in $H_2(X, \partial X)$.*

(1) *If ξ is divisible by an odd prime d, then:*

$$\left|\frac{d^2-1}{2d^2}\xi^2 - \sigma(X) - \sigma_d(K)\right| \leq \dim H_2(X;\mathbb{Z}_d) + 2g.$$

(2) *If ξ is divisible by 2, then:*

$$\left|\frac{\xi^2}{2} - \sigma(X) - \sigma(K)\right| \leq \dim H_2(X;\mathbb{Z}_2) + 2g.$$

The following theorem gives a lower bound for the the genus of a characteritic class embedded in a 4-manifold:

Theorem 3.3 (Acosta [1], Fintushel [8], Yasuhara [27]). *Let X be a smooth closed oriented simply connected 4-manifold with $m = \min(b_2^+(X), b_2^-(X))$ and $M = \max(b_2^+(X), b_2^-(X))$, and assume that $m \geq 2$. If Σ is an embedded surface in X of genus g so that $[\Sigma]$ is characteristic, then*

$$g \geq \begin{cases} \dfrac{\mid \Sigma.\Sigma - \sigma(X) \mid}{8} + 2 - M, & \text{if } \Sigma.\Sigma \leq \sigma(X) \leq 0 \quad \text{or } 0 \leq \sigma(X) \leq \Sigma.\Sigma, \\ \dfrac{9(\mid \Sigma.\Sigma - \sigma(X) \mid)}{8} + 2 - M, & \text{if } \sigma(X) \leq \Sigma.\Sigma \leq 0 \quad \text{or } \leq \Sigma.\Sigma \leq \sigma(X), \\ \dfrac{\mid \Sigma.\Sigma - \sigma(X) \mid}{8} + 2 - m, & \text{if } \sigma(X) \leq 0 \leq \Sigma.\Sigma \quad \text{or } \Sigma.\Sigma \leq 0 \leq \sigma(X). \end{cases}$$

Using the knot filtration on the Heegaard Floer complex $\hat{CF}$, Ozsvath and Szabo introduced in [23] an integer invariant $\tau(K)$ for knots. They showed that $\mid \tau(T(p,q)) \mid = \frac{(p-1)(q-1)}{2}$ (see [23, Corollary 1.7]). In addition, they give a lower bound for the genus of a surface Σ bounding a knot in a 4-manifold. To state their result, let X be a smooth, oriented four-manifold with $\partial X = S^3$ and with $b^+(X) = b_1(X) = 0$. According to Donaldson's celebrated theorem [3], the intersection form of W is diagonalizable. Writing a homology class $[\Sigma] \in H_2(X)$ as $[\Sigma] = s_1.e_1 + \cdots + s_b.e_b$, where e_i are an ortho-normal basis for $H_2(X;Z)$, and $s_i \in \mathbb{Z}$, we can define the L^1 norm of $[\Sigma]$ by $\mid [\Sigma] \mid = \mid s_1 \mid + \cdots + \mid s_b \mid$. Note that this is independent of the diagonalization (since the basis e_i is uniquely characterized, up to permutations and multiplications by ± 1, by the ortho-normality condition). We then have the following bounds on the genus of $[\Sigma]$:

Theorem 3.4 (Ozsvath and Szabo [23]). *Let X be a smooth, oriented four-manifold with $b_2^+(X) = b_1(X) = 0$, and $\partial X = S^3$. If Σ is any smoothly embedded* *surface-with-boundary in* X *whose boundary lies on* S^3, *where it is embedded as the knot* K, *then we have the following inequality*:

$$\tau(K) + \frac{|\,[\Sigma]\,| + [\Sigma].[\Sigma]}{2} \leq g(\Sigma).$$

4. Proof of Statements

To prove Theorems 1.1 and 1.2, we need the following lemma.

Lemma 4.1. *Let* d *be an odd prime number. Then the* d*-signature of a* $(2,q)$*-torus knot* $(q \geq 3)$ *is given by the formula*:

$$\sigma_d((T(2,q)) = -(q-1) + 2\left[\frac{q}{2d}\right],$$

where $[x]$ *denotes the greatest integer less or equal to* x.

Proof. We use Litherland's algorithm to compute $\sigma_d((T(2,q))$. In this case, $y_{1,j} = \frac{1}{p} + \frac{j}{q}$ and $x = \frac{d-1}{2d}$. Therefore,

- $1 + \frac{d-1}{2d} < \frac{1}{2} + \frac{j}{q} < 2$ is equivalent to $1 + \left[\frac{(2d-1)q}{2d}\right] \leq j \leq q-1$.
- $\frac{d-1}{2d} < \frac{1}{2} + \frac{j}{q} < 1 + \frac{d-1}{2d}$ is equivalent to $1 \leq j \leq \left[\frac{(2d-1)q}{2d}\right]$.

Litherland's algorithm yields that $\sigma_d((T(2,q)) = (q-1) - 2\left[\frac{(2d-1)q}{2d}\right]$. It is easy to check that this is equivalent to $\sigma_d((T(2,q)) = -(q-1) + 2\left[\frac{q}{2d}\right]$. □

4.1. *Proof of Theorem* 1.1

Proof.

(1) It is easy to check that $T(-2,3)$ is obtained by a single $(-1,2)$-twisting and also by a single $(-1,3)$-twisting from the unknot, and therefore $T(-2,3)$ is smoothly slice in $\mathbb{C}P^2$, or equivalentely, $g_{\mathbb{C}P^2}(T(-2,3)) = 0$. Theorems 3.2 and 2.1 yield that the only possible degrees are $d \in \{\pm 2, \pm 3\}$; realizable by the twisting disks.

(2) Note that $T(-2,5)$ can be obtained from the unknot by a single $(-1,3)$-twisting (see Fig. 6), which proves that $T(-2,5)$ is smoothly slice in $\mathbb{C}P^2$ with degree $d = +3$ (see [21]). Theorems 3.2 and 2.1 yield that the only possible degrees are $d = \pm 3$; realizable by the twisting disks.

(3) Theorem 2.2 yields that $T(-2,7)$ and $T(-2,9)$ are slice with degree $d = 4$. We can deduce from Theorems 3.2 and 2.1 that the only realizable degrees are $d = \pm 4$.

(4) To show that $g_{\mathbb{C}P^2}(T(-2,11)) = 1$ and $d \in \{\pm 4, \pm 5\}$, we first notice that $T(-2,11)$ is obtained from $T(-2,9)$ by adding two half-twisted bands. By Theorem 2.2, $T(-2,9)$ is smoothly slice in $\mathbb{C}P^2$. Thus $g_{\mathbb{C}P^2}(T(-2,11)) \leq 1$.

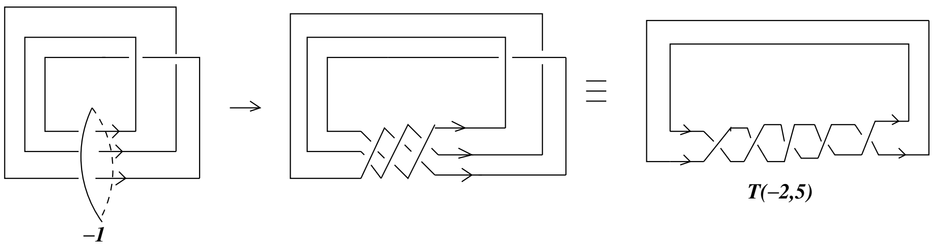

Fig. 6.

To show that $g_{\mathbb{C}P^2}(T(-2,11)) = 1$, let Σ_g be a minimal genus smooth, compact, connected and oriented surface in $\mathbb{C}P^2 - B^4$ with boundary $T(-2,11)$, and assume that $[\Sigma_g] = d\gamma \in H_2(\mathbb{C}P^2 - B^4, S^3, \mathbb{Z})$.

Case 1. If d is even, then by Theorem 3.2.(2), $\mid \frac{d^2}{2} - \sigma(T(-2,11)) - 1 \mid\leq 1 + 2g$. By A.G. Tristram [24], $\sigma(T(-2,11)) = 10$, then d satisfies $20 - 4g \leq d^2 \leq 24 + 4g$. Therefore, $g = 1$ and $d = \pm 4$ are the only possibilities.

Case 2. Assume now that d is odd. We can check that $T(2,11)$ is obtained from the unknot $T(-2,1)$ by a single $(6,2)$-twisting. It was proved in [21] and [7], using Kirby's calculus on the Hopf link [18], that this yields the existence of a properly embedded disk $D \subset S^2 \times S^2 - B^4$ such that $[D] = -2\alpha + 6\beta$ and $\partial D = T(2,11)$. The genus g surface $\Sigma = \Sigma_g \cup D$ satisfies $[\Sigma_g \cup D] = d\gamma - 2\alpha + 6\beta \in H_2(\mathbb{C}P^2 \# S^2 \times S^2, \mathbb{Z})$. Note that Σ is a characteristic class and $[\Sigma]^2 = d^2 - 24$. Assume first that $\mid d \mid\geq 7$, so blowing up $\Sigma \subset S^2 \times S^2 \# \mathbb{C}P^2$ a number of times equal to $d^2 - 24$ gives a genus g surface $\tilde{\Sigma} \subset \mathbb{C}P^2 \# S^2 \times S^2 \# (d^2 - 24)\overline{\mathbb{C}P^2} = X$ (the proper transform) with $[\tilde{\Sigma}]^2 = 0$. If e_i denotes the homology class of the exceptional sphere in the i^{th} blow-up $(i = 1, 2, \ldots, d^2 - 24)$, then $[\tilde{\Sigma}] = d\gamma - 2\alpha + 6\beta - \sum_{i=1}^{i=d^2-24} e_i \in H_2(X, \mathbb{Z})$. The last inequality of Theorem 3.3 yields that $g \geq \frac{|\sigma(X)|}{8}$, which is equivalent to $g \geq \frac{d^2-25}{8}$; which contradicts the assumptions $g \leq 1$ and $\mid d \mid\geq 7$. Therefore, if d is odd then $d \in \{\pm 1, \pm 3, \pm 5\}$ and $g = 1$.

(a) To exclude $d \in \{\pm 1, \pm 3\}$, let Σ_1 be a genus-one smooth, compact, connected and oriented surface in $\mathbb{C}P^2 - B^4$ with boundary $T(-2,11)$, such that $[\Sigma_1] = d\gamma \in H_2(\mathbb{C}P^2 - B^4, S^3, \mathbb{Z})$. Thus, the surface with the other orientation $(\overline{\Sigma}_1, \partial\overline{\Sigma}_1) \subset (\overline{\mathbb{C}P^2} - B^4, S^3)$ is a genus-one surface bounding $T(2,11)$ such that $[\overline{\Sigma}_1] = \pm d\overline{\gamma}$ in $H_2(\overline{\mathbb{C}P^2} - B^4, S^3, \mathbb{Z})$. By Theorem 3.4, we have $\tau(T(2,11)) + \frac{|[\overline{\Sigma}_1]| + [\overline{\Sigma}_1]^2}{2} \leq g(\overline{\Sigma}_1)$. Since $\tau(T(2,11)) = 5, \mid [\overline{\Sigma}_1] \mid=\mid d \mid$ and $[\overline{\Sigma}_1]^2 = -d^2$, then $5 + \frac{|d| - d^2}{2} \leq 1$, a contradiction.

(a) If $d = \pm 5$, then by Lemma 4.1, we have $\sigma_5(T(-2,11)) = 8$ and then Theorem 3.2.(2) yields that $g = 1$ and $d = \pm 5$ are two possibilities. □

4.2. *Proof of Theorem 1.2*

To prove Theorem 1.2, we recall the definition of band surgery:

Band surgery. Let L be a μ-component oriented link. Let $B_1, \ldots, B_\nu$ be mutually disjoint oriented bands in S^3 such that $B_i \cap L = \partial B_i \cap L = \alpha_i \cup \alpha_i'$, where $\alpha_1, \alpha_1', \ldots, \alpha_\nu, \alpha_\nu'$ are disjoint connected arcs. The closure of $L \cup \partial B_1 \cup \cdots \cup \partial B_\nu$ is also a link L'.

Definition 4.2. If L' has the orientation compatible with the orientation of $L - \bigcup_{i=1,\ldots,\nu} \alpha_i \cup \alpha_i'$ and $\bigcup_{i=1,\ldots,\nu}(\partial B_i - \alpha_i \cup \alpha_i')$, then L' is called the link obtained by the *band surgery* along the bands $B_1, \ldots, B_\nu$. If $\mu - \nu = 1$, then this operation is called a *fusion.*

Example 4.3. Let $L_{p,q} = K_1^1 \cup \cdots \cup K_p^1 \cup K_1^2 \cup \cdots \cup K_q^2$ denote the $((p,0),(q,0))$-cable on the Hopf link with linking number 1 (see Fig. 7). Then, $T(2,9)$ can be obtained from $L_{2,4}$ by fusion (see Fig. 8).

Example 4.4. Any $(p, 2kp+1)$-torus knot $(k > 0)$ is obtained from $L_{p,kp}$ by adding $(p-1)(k+1)$ bands (see Yamamoto's construction in [26]). This construction can be generalized to any (p,q)-torus.

For convenience of the reader, we give a smooth surface that bounds $L_{p,q}$ in $T^4 - J$ (J is a 4-ball); due to Kawamura (see [14, 15]): Consider $T^4 = T^2 \times T^2$

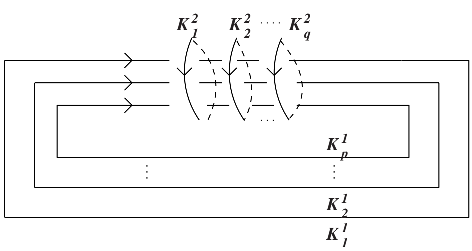

Fig. 7. The link $L_{p,q}$.

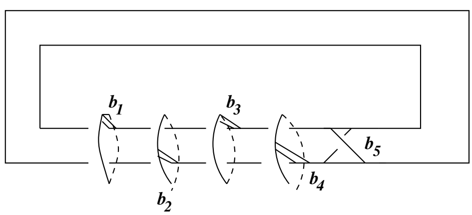

Fig. 8.

where $T^2 = [0,1] \times [0,1]/\sim$ such that $(0,t) \sim (1,t)$ and $(s,0) \sim (s,1)$, and define E and J by:

$$E = \bigcup_{k=1,\ldots,p} \left(\frac{k}{p+1}, \frac{k}{p+1}\right) \times T^2 \cup \bigcup_{k=1,\ldots,q} T^2 \times \left(\frac{k}{q+1}, \frac{k}{q+1}\right)$$

and $J = [\frac{1}{p+2}, \frac{p+1}{p+2}]^2 \times [\frac{1}{q+2}, \frac{q+1}{q+2}]^2$. The 4-ball J contains all self-intersections of E and we have:

Theorem 4.5 (Kawamura [14, 15]). *$\partial(E - J) = E \cap \partial J \subset \partial J$ is the link $L_{p,q}$.*

Auckly proved the following in [6].

Theorem 4.6. *0 is a basic class of T^4.*

To prove Theorem 1.2, we need Proposition 4.7 and Lemma 4.8.

Proposition 4.7. *If $K_{p,q}$ is a knot obtained from $L_{p,q}$ by fusion and Σ_g a smooth, compact, connected and oriented surface properly embedded in $\mathbb{C}P^2 - B^4$ with boundary $K_{p,q}$ in $\partial(\mathbb{C}P^2 - B^4)$. Assume $[\Sigma_g] = d\gamma \in H_2(\mathbb{C}P^2 - B^4, S^3)$, then*

$$2pq - d^2 + \mid d \mid \leq 2(p + q + g) - 2.$$

Proof. By Theorem 4.5, there exists a surface E and a 4-ball J, such that: $\partial(E - J) = L_{p,q}$ (see Fig. 9). Since $K_{p,q}$ is obtained from $L_{p,q}$ by fusion, then there exists a $(p+q+1)$-punctured sphere $\hat{F}$ in $S^3 \times [0,1] \subset J$ such that we can identify this band surgery with $\hat{F} \cap (S^3 \times \{1/2\})$, and $\partial\hat{F} = L_{p,q} \cup K_{p,q}$ with $L_{p,q}$ lies in $S^3 \times \{0\} \cong \partial J \times \{0\}$ and $K_{p,q}$ lies in $S^3 \times \{1\} \cong \partial J \times \{1\}$. The 3-sphere $S^3 \times \{1\} (\cong \partial J \times \{1\})$ bounds a 4-ball $B^4 \subset J$. The surface $F = (E - J) \cup \hat{F}$ is a

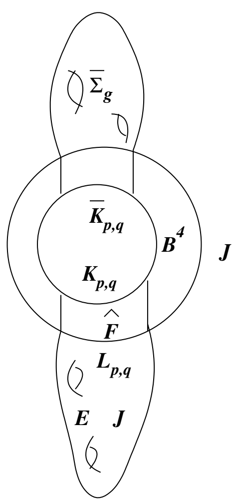

Fig. 9. The surface $\Sigma = (E - J) \cup \hat{F} \cup \overline{\Sigma}_g$.

smooth surface properly embedded in $T^4 - B^4$, and with boundary $K_{p,q}$. Since $K_{p,q}$ bounds a genus g surface $\Sigma_g \subset \mathbb{C}P^2 - B^4$, then $\overline{K}_{p,q}$ bounds a properly embedded genus g surface $\overline{\Sigma}_g \subset \overline{\mathbb{C}P^2} - B^4$ such that $[\overline{\Sigma}_g] = \pm d\overline{\gamma} \in H_2(\overline{\mathbb{C}P^2} - B^4, S^3; \mathbb{Z})$. The smooth surface $\Sigma = F \cup \overline{\Sigma}_g$ in $T^4 \# \overline{\mathbb{C}P^2}$ satisfies $[\Sigma]^2 = F^2 + (\overline{\Sigma}_g)^2$. Since F and E are homologous, then $F^2 = E^2 = 2pq$ which implies that $[\Sigma]^2 = 2pq - d^2$. By Theorem 4.6, 0 is a basic class for T^4, then the basic class of $T^4 \# \overline{\mathbb{C}P^2}$ (the blowup of T^4) is $K = \pm\overline{\gamma}$ (see [9]), and therefore $\mid K.\Sigma \mid = \mid d \mid$. Since $g(E - B^4) = p + q$, then $g(\Sigma) = p + q + g$. The adjunction inequality proved by Kronheimer and Mrowka [22] implies that $[\Sigma]^2 + \mid K.\Sigma \mid \leq 2g(\Sigma) - 2$. Therefore, $2pq - d^2 + \mid d \mid \leq 2(p + q + g) - 2$. ☐

Lemma 4.8. *Let* $(\Sigma_g, \partial\Sigma_g) \subset (\mathbb{C}P^2 - B^4, S^3)$ *be a genus-minimizing smooth, compact, connected and oriented surface properly embedded in* $\mathbb{C}P^2 - B^4$ *with bondary* $T(2, q)$ *and let*

$$[\Sigma_g] = d\gamma \in H_2(\mathbb{C}P^2 - B^4; \mathbb{Z}).$$

(1) *If* d *is even, then* $g = \frac{q-3}{2}$ *and* $d = 0$. *Therefore Conjecture* 1.1 *holds in case* d *is even.*
(2) *Conjecture* 1.1 *holds in case* $d = \pm 1$.

Proof.

(1) For any $q > 0$, we can check that $T(2, q)$ is obtained from $T(2, 3)$ by adding $q - 3$ half-twisted bands, then there is a genus $\frac{q-3}{2}$ cobordism between $T(2, 3)$ and $T(2, q)$. Since $T(2, 3)$ is slice in $\mathbb{C}P^2$, then $g \leq \frac{q-3}{2}$. Since d is even, then by Theorem 3.2(1), $|\frac{d^2}{2} - 1 - \sigma(T(2, q))| \leq 1 + 2g$. By Tristram [24], $\sigma(T(2, q)) = -(q - 1)$, and then $\frac{d^2}{4} + \frac{q-3}{2} \leq g$ which implies that $\frac{q-3}{2} \leq g$ and $d = 0$. Therefore, Conjecture 1.1 holds in case d is even.
(2) To prove that Conjecture 1.1 holds in case $d = \pm 1$, note that $T(2, q)$ is obtained from $L_{(2, \frac{q-1}{2})}$ by fusion, and then apply Proposition 4.7. ☐

Proof of Theorem 1.2. If d is even, then by Lemma 4.8(2), $g_{\mathbb{C}P^2}(T(2, q) = \frac{q-3}{2}$ for $3 \leq q \leq 17$ and the only possible degree is $d = 0$; realizable by the twisting disk Δ. If d is odd, then by Lemma 4.8, we can assume, from now on, that $d \in \mathbb{Z} - \{\pm 1\}$.

(1) If $q = 3$ then it is not hard to check that $T(2, 3)$ can be obtained by a single $(-1, 0)$-twisting from the unknot. This implies that $T(2, 3)$ is smoothly slice in $\mathbb{C}P^2$, or equivalentely $g_{\mathbb{C}P^2}(T(2, 3)) = 0$. To prove that $d = 0$ is the only possibility, let $(\Delta, \partial\Delta) \subset (\mathbb{C}P^2 - B^4, S^3)$ be a smooth 2-disk such that $\partial\Delta = T(2, 3)$, and assume that $[\Delta] = d\gamma \in H_2(\mathbb{C}P^2 - B^4, S^3)$. It is easy to check that $T(2, 1) \overset{(-2,2)}{\longrightarrow} T(-2, 3)$. By [21] and [7], there exists a properly embedded disk $D \subset S^2 \times S^2 - B^4$ such that $[D] = 2\alpha + 2\beta \in H_2(S^2 \times S^2 - B^4, S^3, \mathbb{Z})$ and $\partial D = T(-2, 3)$. The genus g surface $\Sigma = \Sigma_g \cup_{T(2,3)} D$ satisfies

$[\Sigma] = d\gamma + 2\alpha + 2\beta \in H_2(\mathbb{C}P^2\#S^2 \times S^2;\mathbb{Z})$ and then $[\Sigma]^2 = d^2 + 8$. Blowing up Σ a number of times equal to $d^2 + 8$ gives a genus g surface $\tilde{\Sigma} \subset \mathbb{C}P^2\#S^2 \times S^2\#(d^2+8)\overline{\mathbb{C}P^2} = X$ (the proper transform) with $[\tilde{\Sigma}]^2 = 0$. The last inequqlity of Theorem 3.3 yields that $g \geq \frac{d^2+7}{8}$. Therefore, $T(2,3)$ is not slice, a contradiction.

(2) For $q = 5$, note that $T(-2,1) \xrightarrow{(-2,2)} T(-2,5)$. By the same argument as in case $q = 3$, Theorem 3.3 yields that $g \geq \frac{d^2+7}{8}$. This would contradict the assumptions $g \leq 2$ and $\mid d \mid \neq 1$.

(3) For $q = 7$, we can also notice that $T(2,1) \xrightarrow{(-4,2)} T(-2,7)$. By a similar argument, we get a genus g surface $\Sigma = \Sigma_g \cup_{T(2,7)} D$ such that $[\Sigma] = d\gamma + 2\alpha + 4\beta \in H_2(\mathbb{C}P^2\#S^2 \times S^2,\mathbb{Z})$. Since $[\Sigma]^2 = d^2 + 16$, then blowing up Σ a number of times equal to $d^2 + 16$ gives a genus g surface $\tilde{\Sigma} \subset \mathbb{C}P^2\#S^2 \times S^2\#(d^2+16)\overline{\mathbb{C}P^2} = X$ with $[\tilde{\Sigma}]^2 = 0$. The last inequality of Theorem 3.3 yields that $g \geq \frac{d^2+15}{8}$. This would contradict the assumptions $g \leq 2$ and $\mid d \mid \neq 1$.

(4) The case $q = 9$ is similar to $q = 7$ since $T(-2,1) \xrightarrow{(-4,2)} T(-2,9)$, then we can conclude from Theorem 3.3 that $g \geq \frac{d^2+15}{8}$. Since $g \leq 3$, then the only possibilities are $d = \pm 3$ and $g = 3$; excluded by Theorem 3.2(2) and Lemma 4.1 ($\sigma_3(T(2,9)) = -6$).

(5) For $q = 11$, we can check that $T(2,1) \xrightarrow{(-6,2)} T(-2,11)$. By a similar argument, we get a surface Σ such that $[\Sigma] = d\gamma + 2\alpha + 6\beta \in H_2(\mathbb{C}P^2\#S^2 \times S^2;\mathbb{Z})$ and $[\Sigma]^2 = d^2 + 23$. Blowing up Σ a number of times equal to $d^2 + 24$ gives a surface $\tilde{\Sigma} \subset \mathbb{C}P^2\#S^2 \times S^2\#(d^2+24)\overline{\mathbb{C}P^2} = X$ such that $[\tilde{\Sigma}] = d\gamma + 2\alpha + 6\beta - \sum_{i=1}^{i=d^2+24} e_i \in H_2(X,\mathbb{Z})$ and then $[\tilde{\Sigma}]^2 = 0$. Since $\sigma(X) = -d^2 - 23$, then Theorem 3.3 implies that $g \geq \frac{d^2+23}{8}$. Since $g \leq 4$, then the only possibilities are $d = \pm 3$ and $g = 4$; excluded by Theorem 3.2(2) and Lemma 4.1 ($\sigma_3(T(2,11)) = -8$).

(6) For $q = 13$, we can easily check that $T(2,-1) \xrightarrow{(-6,2)} T(-2,13)$, and Lemma 4.1 yields that $\sigma_3(T(2,13)) = -8$. Then, the argument is similar to the case $q = 11$.

(7) For $q = 15$, we have $T(2,-1) \xrightarrow{(-8,2)} T(-2,15)$. Theorem 3.3 implies that $g \geq \frac{d^2+31}{8}$; which excludes the cases where $\mid d \mid \geq 5$. Lemma 4.1 yields that $\sigma_3(T(2,15)) = -10$; which yields that the case $d = \pm 3$ and $g = 5$ are two possibilities.

(8) For $q = 17$, we have $T(2,-1) \xrightarrow{(-8,2)} T(-2,17)$. Lemma 4.1 yields that $\sigma_3(T(2,17)) = -12$. Then the argument is similar to the case $q = 15$.

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