Hyperbolic knots and twiting

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ABSTRACT

We classify knots abtained from twisting prime reduced links, and give a new condition for recognition of hyperbolic knots. We also give a combinatorial condition to exclude torus knots.

1. Introduction

Let $L = K \cup C_1 \cup C_2 ... \cup C_t$ be a prime augmented link (see [8] for definition) where K is the unknot and $\{C_1, C_2, ..., C_t\}$ is a disjoint union of trivial circles such the family $\{C_i\}_{i=1}^{i=t}$ bounds non-parallel disjonts disks $\{D_i\}_{i=1}^{i=t}$, each of which is perpendicular to the projection plane and intersects the knots at exactly m points i.e.$\mid D_i \cap K \mid = m \geq 2$ for any $i = 1, 2, ...t$. For convenience, we denote by $n = (n_1, n_2, ..., n_t)$ and $D = (D_1, ..., D_t)$. We successively perform a series of full n_i-twisting ($i \in \{1, 2, .., t\}$)along Dehn disks $D_1, D_2, ..., D_t$:

$$K \overset{n_1-twist}{\longrightarrow} K(n_1) ... \overset{n_k-twist}{\longrightarrow} K(n_1, ..., n_k) = K_{D,n}.$$

By Thurston geometrization theorem $M^3 = S^3 - intN(K \cup C_1 \cup C_2 ... \cup C_t)$ is either Seifert fibred, or toroidal, or hyperbolic. In case $t = 1$ and $\mid n \mid > 1$, we proved in [2] that the resulting knot $K_{D,n}$ is of the same type as the pair (K, D). If (K, D) is hyperbolic and $K_{D,n}$ is toroidal, then $\mid n \mid = 1$ (Ait Nouh-Matignon-Motegi [2] and Gordon-Luecke [7]). In this paper, we show a similar result provided that $\mid D_i \cap K \mid = m \geq 2$ for all $i = 1, 2, ...t$.

Theorem 1.1. *Assume that $\mid n_i \mid > 1$ for any $i = 1, 2, ...t$, and L is prime and augmented. If $\mid D_i \cap K \mid = m \geq 2$ for any $i = 1, 2, ...t$, then $K_{D,n}$ is of the same type as (L, D). More precisely,*

(1) *If M^3 is Seifert fibred, then $t = 1$, and $K_{D,n}$ is an $(m, n_1 m \pm 1)$-torus knot.*

(2) *If M^3 is toroidal, then $K_{D,n}$ is satellite.*

(3) *If M^3 is hyperbolic, then $K_{D,n}$ is hyperbolic.*

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Corralary 1.1. *If (i) $m \geq 3$ is odd, and (ii) $t \geq 2$, (iii) $|\, n_i \,| > 1$ for any $i = 1, 2, ...t$, and (iv) any incoming arc of K to D_i forms a clasp with $C_i = \partial D_i$ (see Figure 2.(i)) then both L and $K_{D,n}$ are hyperbolic.*

In the following Corollary, we assume that $|\, D_i \cap K \,| = 2$ for $i \in \{1, 2, ..., k\}$.

Corralary 1.2. *Let k be a knot with a prime, twist-reduced diagram $D(k)$. Assume that $D(k)$ has $tw(D) \geq 2$ twist regions, and that each region contains at least 4 crossings. Then k is not a torus knot.*

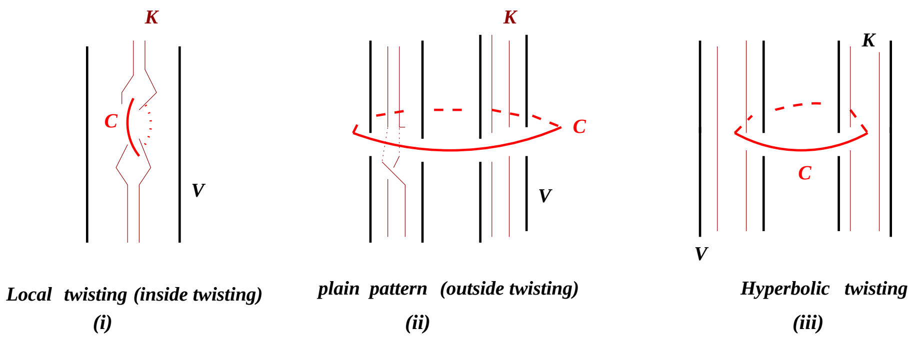

Figure 1:

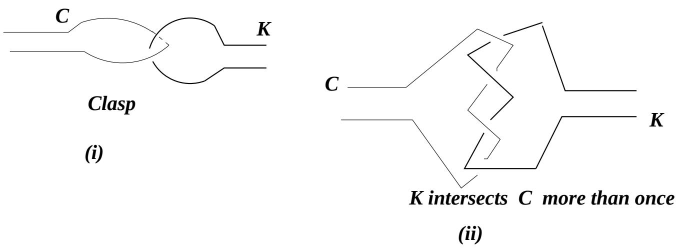

Figure 2:

2. Preliminaries

To prove Theorem 1.1., we need the following theorems:

Theorem 2.1. (D. Gabai [6]) *Let M^3 be a Haken 3-manifold such that ∂M^3 contains a torus component denoted T. Denote by $M_\phi = M \bigcup_{\phi} S^1 \times D^2$ where $\phi \in \mathbb{Q}$, the manifold obtained by ϕ-Dehn filling along T. Let S be a minimizing surface for the Thurston norm in $H_2(M^3, \partial M^3, \mathbb{Z})$. Then $x(\phi(S))$ is decreasing for at most one slope.*

Theorem 2.1. (D. Gabai [5], J. Berge [4]) *Let $V = S^1 \times D^2$ and k a knot in V such that k is not contained in a 3-ball of V, and $\alpha = \dfrac{p}{q} \in \mathbb{Q} \cup \{\pm\infty\}$. Then $V(\alpha) \cong V$ if and only if k is a 0-bridge or a 1-bridge relative to V.*

3. Proof of results

Proof of Theorem 1.1.

If M^3 is not hyperbolic, then M^3 is either Seifert fibred or toroidal.

(1) If M^3 is Seifert fibred, then $S^3 - intN(K \cup C_1 \cup C_2 ... \cup C_t)$ is a $(1, p)$-fibred solid torus in which K is a regular fiber. Hence $K_{D,n}$ is a $(p, (\sum_{i=1}^{i=t} n_i)p \pm 1)$-torus knot in S^3. Since the wrapping number $\mid D_i \cap K = m \mid$ for any $i = 1, 2, ..., t$, and L is reduced, then $t = 1$ and $p = m$.

(2) If the twisting is toroidal, then M^3 contains an essential torus T^2.

By the *solid torus theorem* (J.W. Alexander [3]), T^2 bounds a solid torus V in S^3. From now on, we denote $C = C_1 \cup C_2 ... \cup C_k$, There are five cases (see Figure 2):

Case 1**:** $K \cup C \subset S^3 - V$

This case is excluded since T^2 would be compressible (the meridian disk of V would be a compressing disk of T^2).

Case 2. $K \cup C \subset V$ (Local twisting (see Figure 2(i))

Since T^2 is incompressible, then V is knotted. C_1 is trivial in V, then C_1 is equivalent to a meridian of V. Therefore, C_1 is a 0-bridge knot. By Berge-Gabai's theorem $V(C_1, -\dfrac{1}{n_1}) = V$. Let $N_1^3 = V - int(K \cup C_1)$. Since K is a trivial knot in V, then the winding number $wind_V(K) = 0$. Therefore, there exist a surface $(S_1, \partial S_1) \subset (N_1^3, \partial V)$ such that $\partial S_1 = m_V$ where m_V is a meridian of ∂V. Let $[(S_1, \partial S_1)] = z \in H_2(N_1^3, \partial N_1^3, \mathbb{Z})$. Since N_1^3 is ∂-irreducible, then z is not a disk and therefore its Thurston norm $x(z) \neq 0$. If we denote by $N_1^3(\alpha) = N_1^3 \bigcup_{\phi_\alpha} N(C_1)$, then the trivial surgery along C_1 gives $x(\phi_{\frac{1}{0}}(z)) = 0$, and by Gabai's theorem: $x(\phi_{-\frac{1}{n_1}}(z)) = x(z)$. This implies that $x(\phi_{-\frac{1}{n_1}}(z)) \neq 0$, and then $wrap_V(K_{n_1}) \geq 2$. Therefore, K_{n_1} is a satellite knot, for any $n_1 \neq 0$.This proves that $\partial V(C_1, -\dfrac{1}{n_1}) = \partial V$ is essential in $M(-\dfrac{1}{n_1}) \cong E(K_{n_1})$.

Now let $N_2^3 = N_1^3 - intN(C_2)$ $(= V - intN(K \cup C_1 \cup C_2))$, which is Haken. $wind_V(K) = 0$, then $wind_V(K_{D_1,n_1}) = 0$. Therefore, there exist a surface $(S_2, \partial S_2) \subset (N_2^3, \partial V)$ such that $\partial S_2 = m_V$ where m_V is a meridian of ∂V. By a similar argument as above, we can re-apply Gabai's theorem to show that $K(n_1, n_2)$ is a satellite knot, and $M(-\dfrac{1}{n_1})(-\dfrac{1}{n_2}) \cong E(K(n_1, n_2))$. By induction, we can show that $K_{D,n}$ is satellite.

Case 3: $K \subset V$ and $C \subset S^3 - V$ (T^2 is separating) (plain pattern twisting (see Figure 2(i))

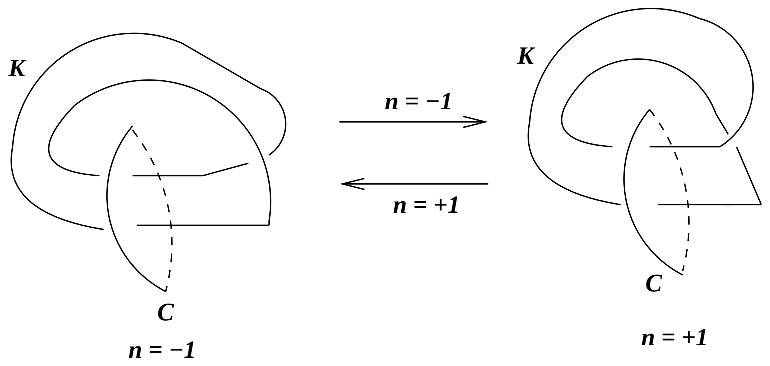

Figure 3:

K is trivial, then V is an unknotted torus, and then its core ℓ is a trivial knot. Since $\mid n_1 \mid > 1$, then the pair (C, ℓ) is not Mathieu-Domergue exception (see Figure 3), then ℓ_{n_1} is knotted. Proving that $V(C_1, -\dfrac{1}{n_1}) = N(\ell_{n_1})$ is equivalent to $S^3 - V(C_1, -\dfrac{1}{n_1}) = E(\ell_{n_1})$. Let $W = S^3 - intV$ which is an unknotted torus containing C_1. Since $K \subset V$, then $K \subset S^3 - intW$.

$$\begin{aligned}
E(\ell_{n_1}) &= (W - intN(\ell \cup C_1)) \bigcup_{\phi_{-\frac{1}{n_1}}} N(C_1) \\
&= (S^3 - intN(\ell)) - intN(C_1) \bigcup_{\phi_{-\frac{1}{n_1}}} N(C_1) \\
&= (W - intN(C)) \bigcup_{\phi_{-\frac{1}{n_1}}} N(C_1) \\
&= W(C_1, -\frac{1}{n_1})
\end{aligned}$$

"In other words, the surgery is made outside V"

$S^3 = V \cup W$ is a genus two Heegard splitting of S^3.Therefore, we have $S^3 = V \cup W$ which implies that $S^3(C_1, -\dfrac{1}{n_1}) = V(C_1, -\dfrac{1}{n_1}) \cup W(C_1, -\dfrac{1}{n_1})$. Then $S^3 = V(C_1, -\dfrac{1}{n_1}) \cup E(\ell_{n_1})$, which implies that$V(C_1, -\dfrac{1}{n_1}) = N(\ell_{n_1})$. To prove that $K_{n_1} \subset N(\ell_{n_1})$, let $V_1 = V - intN(K)$, then we have:

$$\begin{aligned} S^3 = V \cup W &\implies S^3 - intN(K) = (V - intN(K)) \cup W \\ &\implies S^3 - intN(K) = V_1 \cup W \\ &\implies M(-\frac{1}{n_1}) = V_1(C_1, -\frac{1}{n_1}) \cup W(C_1, -\frac{1}{n_1}) \\ &\implies E(K_{n_1}) = V_1(C_1, -\frac{1}{n_1}) \cup E(\ell_{n_1}) \\ &\implies E(\ell_{n_1}) \subset E(K_{n_1}) \\ &\implies K_{n_1} \subset N(\ell_{n_1}) \end{aligned}$$

Let's prove now that $wr_{N(\ell_{n_1})}(K_{n_1}) \geq 2$, we note that K_{n_1} is obtained by n_1-twisting from K along C_1, then K is obtained by $(-n_1)$-twisting from K_{n_1}. Therefore, K_{n_1} is not contained in a 3-ball of $V_1(C_1, -\frac{1}{n_1})$ and therefore $wrap_{N(\ell_{n_1})}(K_{n_1}) \geq 2$.

Case 4**:** $C \subset V$ and $K \subset S^3 - V$.

V is standardly embeddded. Note that $K \subset W$ and $C \subset S^3 - W$. V and W plays similar roles, then we apply the argument of Case 1. Therefore, ∂W_{n_1} is essential in $M(-\frac{1}{n_1}) = E(K_{n_1})$, or equivalentely ∂V_{n_1} is essential in $M(-\frac{1}{n_1}) = E(K_{n_1})$. Consequentely, K_{n_1} is a satellite knot with companion ℓ_{n_1}.

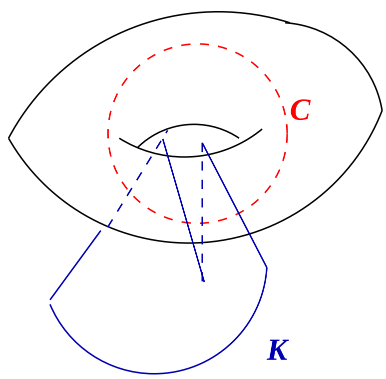

Figure 4: Case 4

The argument can be repeated recursively for $i \in \{1, 2, ..., k\}$. Therefore, Theorem 1.1 holds.

Case 5**:** The case where some $C_i \subset V$ and some $C_i \subset S^3 - V$, yield always a satellite knot (C_i does not intersect ∂V). The argument is similar to the above cases.

Proof of Corrolary 1.1**:**

Since $t \geq 2$, then case (1) of Theorem 1.1 is excluded. Since $m \geq 3$ is odd and K and C_i form a clasp then case (2) of Theorem 1.1 is excluded. Therefore, L and $K_{D,n}$ are both hyperbolic.

Proof of Corrolary 1.2**:**

Since each twist region has at least 4 crossing, then let $c_i \geq 4$ be the number of crossing in the i-th region R_i, with crossing circle C_i. The *augmented link* L is obtained by performing $\frac{1}{n_i}$-Dehn surgery on each C_i as follows: If c_i is even (resp. odd), then $n_i = -\epsilon_i c_i$ (resp. $n_i = -\epsilon_i \dfrac{c_i - 1}{2}$), where ϵ_i is the sign of the twisting. Then $L = C \cup K$, where K is the image of k after performing Dehn surgeries along the crossing circles. Note that K is a knot in S^3.

Case 1. If K is the unknot, then $k = K_{D,n}$. Since $c_i \geq 4$ then $\mid n_i \mid \geq 2$. Therefore, Theorem 1.1 implies that k is either a $(2, 2n_1 \pm 1)$-torus knot, or a satellite knot, or a hyperbolic knot. Since $tw(k) \geq 2$, then k is not a $(2, 2n_1 \pm 1)$-torus knot, and therefore k is not a torus knot.

Case 2. If K is knotted, then by construction, the unknotting number of K is less or equal to the number of twist regions with odd number of crossings. Provided that we interchange $n_i = -\epsilon_i \dfrac{c_i - 1}{2}$ and $n_i = -\epsilon_i \dfrac{c_i + 1}{2}$, the knot K can be transformed to the unknot K_0, and then apply Theorem 1.1.

Remark. Note that the link $L_0 = C \cup K_0$ obtained in Case 2 is not the augmented link, and that $S^3 - L \cong S^3 - L_0$. In particular, since $S^3 - L$ is hyperbolic (Adams, Agol,Thurston), then $S^3 - L_0$ is hyperbolic.

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